Solutions to Homework 8, Math 310

We will show that all the relations below are equivalence relations associated to suitable functions and then the rest will be obvious. So, we will just write down these functions. Recall from class that if \( f: A \to B \) is a function, we can define a relation on \( A \) by saying that \( a \sim a' \) for \( a, a' \in A \) if \( f(a) = f(a') \). This is easily seen to be an equivalence relation and the set of equivalence classes can be identified via \( f \) to the set \( f(A) \), the image of \( f \).

1. Define a relation on \( \mathbb{R} \) as follows. Two real numbers \( x, y \) are equivalent if \( x - y \in \mathbb{Z} \). Show that this is an equivalence relation. Show that the set of equivalence classes of this relation is (naturally) bijective to the set of points on the unit circle.

   \[ \text{Proof. Define } \phi: \mathbb{R} \to \mathbb{R}^2 \text{ by } \phi(x) = (\cos 2\pi x, \sin 2\pi x). \]

2. Define a relation on \( P = \mathbb{R}^2 - \{(0, 0)\} \) as follows. If \((a, b), (c, d) \in P\), they are related if for some positive real number \( \alpha \), \( a = \alpha c \) and \( b = \alpha d \). Show that this is an equivalence relation and the set of equivalence classes is (naturally) bijective to the points on a unit circle.

   \[ \text{Proof. Define } \phi: P \to \mathbb{R}^2 \text{ by } \phi(x, y) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right). \]

3. Define a relation on \( \mathbb{R}[x] \), the set of polynomials as follows. Two polynomials \( f(x), g(x) \) are related if \( f(0) = g(0) \). Show that this is an equivalence relation and the set of equivalence classes is (naturally) bijective to \( \mathbb{R} \).

   \[ \text{Proof. Define } \phi: \mathbb{R}[x] \to \mathbb{R} \text{ by } \phi(f(x)) = f(0). \]

4. Let \( C(\mathbb{R}) \) be the set of all continuous functions on \( \mathbb{R} \). If \( f, g \in C(\mathbb{R}) \) we say that \( f \sim g \) if \( \int_0^1 f dx = \int_0^1 g dx \). Show that this is an equivalence relation. Can you identify the set of equivalence classes with a familiar set?

   \[ \text{Proof. Define } \phi: C(\mathbb{R}) \to \mathbb{R} \text{ by } \phi(f) = \int_0^1 f dx. \]

5. Define a relation on \( \mathbb{N} \times \mathbb{N} \) by declaring that \((a, b) \) is related to \((c, d) \) if \( a + d = b + c \). Show that this is an equivalence relation and the set of equivalence classes is (naturally) bijective to \( \mathbb{Z} \).

   \[ \text{Proof. Define } \phi: \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \text{ by } \phi(a, b) = a - b. \]

6. Let \( P = \mathbb{Z} \times (\mathbb{Z} - \{0\}) \), the set of pairs of integers \((a, b)\) with \( b \neq 0 \). Define a relation on \( P \) by declaring that for \((a, b), (c, d) \in P\), \((a, b) \sim (c, d) \) if \( ad = bc \). Show that this is an equivalence relation.
relation and the set of equivalence classes is (naturally) bijective to \( \mathbb{Q} \).

**Proof.** Define \( \phi : P \to \mathbb{Q} \) by \( \phi(a, b) = \frac{a}{b} \). \( \square \)

(7) Define a function \( \phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) as follows.

\[
\phi(m, n) = \frac{(m + n - 2)(m + n - 1)}{2} + n.
\]

Show that \( \phi \) is a bijection.

**Proof.** There are several ways of arguing this. I will construct an inverse function to \( \phi \). As usual we will use the notation \( \binom{a}{2} \) to mean \( \frac{a(a-1)}{2} \) for any \( a \in \mathbb{N} \). Thus,

\[
\phi(m, n) = \left( \frac{m + n - 1}{2} \right) + n.
\]

For any \( n \in \mathbb{N} \), define the set \( S(n) = \{ a \in \mathbb{N} \mid \binom{a}{2} \geq n \} \). This set is clearly nonempty—for example, \( 2n \in S(n) \). So, by induction, \( S(n) \) has a minimal element, which we denote by \( a_n \). So, we get,

\[
\binom{a_n}{2} \geq n > \binom{a_n - 1}{2} \quad (1)
\]

This yields,

\[
\binom{a_n}{2} - \binom{a_n - 1}{2} \geq n - \binom{a_n - 1}{2} > 0
\]

After simplifying, this becomes,

\[
a_n > n - \binom{a_n - 1}{2} \geq 1 \quad (2)
\]

Define \( \psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) by the formula,

\[
\psi(n) = \left( a_n + \frac{a_n - 1}{2} - n, n - \frac{a_n - 1}{2} \right) \quad (3)
\]

By equation 2, notice that this is indeed a function to \( \mathbb{N} \times \mathbb{N} \). Next we check that this is the inverse to \( \phi \). For \( (m, n) \in \mathbb{N} \times \mathbb{N} \), let \( N = \phi(m, n) \). Since \( \frac{(m+n-1)}{2} = \phi(m, n) - n < N \), we see that by definition, \( a_N > m + n - 1 \). We claim that \( a_N = m + n \).
Clearly, it suffices to show that \((\frac{m+n}{2}) \geq N\).

\[
\begin{align*}
\binom{m+n}{2} &= \left[ \binom{m+n}{2} - \binom{m+n-1}{2} \right] + \binom{m+n-1}{2} \\
&= (m+n-1) + \phi(m,n) - n \\
&= (m-1) + N \geq N
\end{align*}
\]

Thus, we get,

\[
\psi \circ \phi(m, n) = \psi(N)
\]

\[
= \left( m + n + \binom{m+n-1}{2} \right) - N, N - \binom{m+n-1}{2}
\]

\[
= (m + n - n, n) = (m, n)
\]

So, we get \(\psi \circ \phi = \text{Id}_{\mathbb{N} \times \mathbb{N}}\). Similarly,

\[
\phi \circ \psi(n) = \phi\left( a_n + \binom{a_n-1}{2} - n, n - \binom{a_n-1}{2} \right)
\]

\[
= \left( a_n - 1 \right) + n - \binom{a_n-1}{2} = n
\]

Thus, \(\phi \circ \psi = \text{Id}_\mathbb{N}\). This proves the result.

\[
\square
\]

Proof. Another way of proving this is as follows. As usual, we will show that \(\phi\) is injective and surjective. To show injectivity, we proceed as follows. We claim first that if \(m + n > m' + n'\), then \(\phi(m, n) > \phi(m', n')\). For this, clearly, it suffices to prove the case when \(m + n = m' + n' + 1\), by induction. In this case, we get,

\[
\phi(m, n) - \phi(m', n') = \left( \frac{m+n-1}{2} \right) + n - \left( \frac{m'+n'-1}{2} \right) - n'
\]

\[
= \left( \frac{m'+n'}{2} \right) - \left( \frac{m'+n'-1}{2} \right) + n - n'
\]

\[
= m' + n' - 1 + n - n' = m' + n - 1 > 0
\]

Thus, if \(\phi(m, n) = \phi(m', n')\), we must have \(m + n = m' + n' = p\). But, then we get,

\[
\binom{p-1}{2} + n = \phi(m, n) = \phi(m', n') = \binom{p-1}{2} + n',
\]

and thus, \(n = n'\). Since \(m + n = m' + n'\), we get \(m = m'\), proving injectivity.
To show surjectivity, let $n \in \mathbb{N}$. Then, as in the previous proof, we can define $a_n \in \mathbb{N}$, so that inequality 1 is satisfied. Then one takes,

$$(p, q) = \left( a_n + \left( \frac{a_n - 1}{2} \right) - n, n - \left( \frac{a_n - 1}{2} \right) \right)$$

and checks that $\phi(p, q) = n$, proving surjectivity. \qed