Solutions to Midterm, Math 310

PART A

We will assume that our present knowledge for Part A is just the following. We have the natural numbers or counting numbers, denoted by the letter \( \mathbb{N} \). These are just the collection \( \{1, 2, 3, \ldots\} \). It has operations + and \( \times \) called addition and multiplication with the basic properties of closure, commutativity, associativity and distributivity. One may also write for \( a, b \in \mathbb{N} \), \( a \times b = a \cdot b = ab \) interchangeably.

**Definition 1.** An element \( a \in \mathbb{N} \) is called a perfect square if there exists a \( b \in \mathbb{N} \) such that \( a = b^2 = b \cdot b \).

1. Give a direct proof to show that if \( a, b \in \mathbb{N} \) are perfect squares, so is \( ab \).

   **Proof.** By definition, \( a = m^2, b = n^2 \) for \( m, n \in \mathbb{N} \). Then \( ab = m^2n^2 = (mn)^2 \) and since \( mn \in \mathbb{N} \), by definition \( ab \) is a perfect square. \( \Box \)

2. Write the truth table for the statement \( P \Rightarrow (Q \land R) \) where \( P, Q, R \) are mathematical statements.

   **Proof.**

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   \( \Box \)

3. Write the negation of the following statement in English without using quantifier symbols.

   \( \forall a \in \mathbb{N}, \ a \cdot 1 = a. \)

   **Proof.** There is a natural number \( a \) such that \( a \cdot 1 \neq 1 \). \( \Box \)

4. Write the converse and contrapositive of the statement where \( a, b \in \mathbb{N} \): If \( a \) is a perfect square so is \( ab^2 \).

   **Proof.** The converse is: If \( ab^2 \) is a perfect square then so is \( a \). The contrapositive is: If \( ab^2 \) is not a perfect square nor is \( a \). \( \Box \)
(5) For any two sets $A, B$, show that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

Proof. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Thus $x \in A$ or $x \in B$ but $x$ is not in both $A$ and $B$. Thus if $x \in A$, then $x \notin B$ and thus $x \in A - B$. If $x \in B$, then $x \notin A$ and thus $x \in B - A$. Thus $x$ is either in $A - B$ or $B - A$ and thus $x \in (A - B) \cup (B - A)$, showing that $(A \cup B) - (A \cap B) \subset (A - B) \cup (B - A)$.

Next, let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$. Thus, either $x \in A$ but not in $B$ or $x \in B$ but not in $A$. Thus $x \in A \cup B$, but $x \notin A \cap B$. So, $x \in (A \cup B) - (A \cap B)$ showing the reverse inclusion and thus the equality of sets. 

(6) Define injective functions and composition of functions. Show that if $f : A \to B$ is injective and $g : B \to C$ is injective, then $g \circ f : A \to C$ is injective.

Proof. I will not repeat the definitions. Given $f, g$ as above, to show injectivity of $g \circ f$, we need to show that if $a, a' \in A$ and $g \circ f(a) = g \circ f(a')$, then $a = a'$. Let $f(a) = b, f(a') = b'$. Then the above equation yields $g(b) = g(b')$ and thus by injectivity of $g$, we see that $b = b'$. This means $f(a) = f(a')$ and by injectivity of $f$ we get $a = a'$ proving the result. 

PART B

(7) Prove by induction on $n \in \mathbb{N}$ that for any $a \in \mathbb{Z}$, $(a + 2)^n - a^n$ is even, where as usual, an integer $m$ is even, if it is equal to $2n$ for some integer $n$.

Proof. Let $P(n)$ be the statement that $(a + 2)^n - a^n$ is even. Then $P(1)$ is just the statement $(a + 2) - a = 2$ is even. Since 2 is even, we see that $P(1)$ is true.

Next we show that $P(n) \Rightarrow P(n + 1)$. So, assume that $P(n)$ is true. So, 

$$(a + 2)^n - a^n = 2m$$

for some integer $m$. Then,

$$(a + 2)^{n+1} - a^{n+1} = (a + 2)((a + 2)^n - a^n) + (a + 2)a^n - a^{n+1}$$

$$= 2m(a + 2) + a^{n+1} + 2a^n - a^{n+1}$$

$$= 2(am + 2m + a^n).$$

Since $am + 2m + a^n$ is an integer, we see that $P(n + 1)$ is true. Thus by induction, $P(n)$ is true for all $n \in \mathbb{N}$. 

□
(8) Calculate the remainder of 1753² when divided by 20. (Hint: Calculate the remainder of 1753 when divided by 10).

Proof. We can write 1753 = q × 10 + 3 for some integer q. (In fact q = 175, but that will be irrelevant). Thus,

\[ 1753^2 = (10q + 3)^2 \]
\[ = 100q^2 + 60q + 9 \]
\[ = 20 \times (5q^2 + 3q) + 9 \]

So, the remainder of 1753² divided by 20 is 9. □

I know that some of you will just write the square of 1753 and divide by 20 to get the remainder. But, what if I asked you to the above with 1753⁴ or 1753⁸ or whatever? So, good to learn to separate what is relevant and what is not.

(9) If \( n \in \mathbb{N} \) is odd and \( a, b \in \mathbb{Z} \), show that there exists an integer \( m \) so that \( a^n + b^n = m(a + b) \), using induction.

Proof. Since odd natural numbers can be expressed as \( 2n - 1 \) for \( n \in \mathbb{N} \) what we want to show is that for any \( n \in \mathbb{N} \), \( a^{2n-1} + b^{2n-1} = m(a + b) \) for some integer \( m \). As usual, let \( P(n) \) be the statement that \( a^{2n-1} + b^{2n-1} = m(a + b) \) for some integer \( m \). We wish to show that \( P(n) \) is true for all \( n \).

\( P(1) \) is just the statement that \( a + b = m(a + b) \) for some integer \( m \). Clearly, we can take \( m = 1 \) and thus \( P(1) \) is true.

Next, we show that \( P(n) \Rightarrow P(n+1) \). So, assume that \( P(n) \) is true. Thus, we have an equation,

\[ a^{2n-1} + b^{2n-1} = m(a + b), \]

for some integer \( m \). Now,

\[ a^{2n+1} + b^{2n+1} = a^2(a^{2n-1} + b^{2n-1}) + b^{2n+1} - a^2b^{2n-1} \]
\[ = ma^2(a + b) + b^{2n-1}(b^2 - a^2) \]
\[ = ma^2(a + b) + b^{2n-1}(b - a)(b + a) \]
\[ = (ma^2 + b^{2n-1}(b - a))(a + b). \]

Since \( ma^2 + b^{2n-1}(b - a) \) is an integer, we see that \( P(n+1) \) is true and thus proving our result by induction. □