We will assume that our present knowledge is just the following.

We have the natural numbers or counting numbers, usually denoted by the letter \( \mathbb{N} \). These are just the collection \( \{1, 2, 3, \ldots\} \). These have the following basic properties. Lower case English letters will denote natural numbers in the following.

- **Closure**: We may add two natural numbers to get a natural number. Addition is denoted by the symbol \(+\). Similarly we can multiply two natural numbers to get a natural number. Multiplication is denoted by either \( \cdot \) or just writing the numbers next to each other.
- **Commutativity**: \( a + b = b + a \) and \( ab = ba \) for all natural numbers \( a, b \).
- **Associativity**: \( a + (b + c) = (a + b) + c \) and \( a(bc) = (ab)c \) for all natural numbers \( a, b, c \).
- **Distributivity**: \( a(b + c) = ab + ac \) for all natural numbers \( a, b, c \).
- **Notation**: In this notation, we have \( 2 = 1 + 1 \), \( 3 = 2 + 1 \) etc. and all these are different.

(1) We start with a definition.

**Definition 1.** A natural number \( a \) is called even if there exists another natural number \( b \) such that \( a = 2b \). A natural number \( a \) is called odd if \( a + 1 \) is even.

Write a know-show table and a proof for the following two theorems.

**Theorem 1.** If \( a \) is even, then \( a + 2 \) is even, where \( a \) is a natural number.

**Theorem 2.** If \( a \) is even, then \( 3a^2 + 4a + 7 \) is odd where \( a \) is a natural number and the notation \( a^2 \) as usual stands for \( a \cdot a \).

(2) With our present knowledge, can you prove that any natural number is either even or odd? If you can, write a proof. If you cannot, explain why.

(3) Recall from class the following definition.

**Definition 2.** If \( a, b \) are natural numbers we say \( a \) is greater than \( b \), written \( a > b \), if there exists a natural number \( k \) with \( a = b + k \).

Write a know-show table and a proof for the following theorem.
Theorem 3. If $a, b$ are natural numbers and $a > b$, then $a^2 + b^2 > 2ab$.

(4) With our present knowledge, can you prove that if $a, b$ are any two natural numbers, then exactly one of the following must occur:

$a = b$ or $a > b$ or $b > a$?

If you can, write a proof and if you cannot, explain why.