Homework 1, Math 310, due Friday 4, September

We will assume that our present knowledge is just the following.

We have the *natural numbers* or *counting numbers*, usually denoted by the letter $\mathbb{N}$. These are just the collection $\{1, 2, 3, \ldots\}$. These have the following basic properties. Lower case English letters will denote natural numbers in the following.

- **closure**: We may add two natural numbers to get a natural number. Addition is denoted by the symbol $+$. Similarly we can multiply two natural numbers to get a natural number. Multiplication is denoted by either a $\cdot$ or just writing the numbers next to each other.
- **commutativity**: $a + b = b + a$ and $ab = ba$ for all natural numbers a, b.
- **associativity**: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ for all natural numbers $a, b.c$.
- **distributivity**: $a(b+c) = ab+ac$ for all natural numbers a, b, c.
- **Notation**: In this notation, we have $2 = 1 + 1$, $3 = 2 + 1$ etc. and all these are different.

(1) We start with a definition.

Definition 1. A natural number a is called even if there exists another natural number b such that $a = 2b$. A natural number a is called odd if $a + 1$ is even.

Write a know-show table and a proof for the following two theorems.

Theorem 1. *If a is even, then $a + 2$ is even, where a is a natural number.*

Theorem 2. *If a is even, then $3a^2 + 4a + 7$ is odd where a is a natural number and the notation a^2 as usual stands for $a \cdot a$.*

(2) With our present knowledge, can you prove that any natural number is either even or odd? If you can, write a proof. If you cannot, explain why.

(3) Recall from class the following definition.

Definition 2. If a, b are natural numbers we say a is greater than b, written $a > b$, if there exists a natural number k with $a = b + k$.

Write a know-show table and a proof for the following theorem.

Theorem 3. *If a, b are natural numbers and $a > b$, then $a^2 + b^2 > 2ab$.*

(4) With our present knowledge, can you prove that if a, b are any two natural numbers, then exactly one of the following must occur:

$$a = b \quad \text{or} \quad a > b \quad \text{or} \quad b > a?$$

If you can, write a proof and if you can not, explain why.