Homework 10, Math 310, due 16th November 2009

When proving a sequence $\{x_n\}$ is a CS, the steps must be always the same. Start with an $\epsilon > 0$, which you are not allowed to choose. Then you must say what $N \in \mathbb{N}$ you will choose. This could be a desription depending on ϵ and if it is not obvious such an N exists, you must justify it. Then for this N, you must check that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

(1) Let $\{x_n\}$ be a sequence and define a new sequence $\{y_n\}$ by the formula, $y_k = 10^k$ for $1 \leq k \leq 100$ and $y_{100+k} = x_k$ for $k \in \mathbb{N}$. (So, $y_1 = 10, y_2 = 100, y_{101} = x_1, y_{102} = x_2$ etc.) Show that $\{x_n\}$ is a CS if and only if $\{y_n\}$ is a CS.

(2) Let $\{x_n\}$ be a sequence and define a new sequence $\{y_n\}$ by the formula, $y_k = x_{2k}$ for all $k \in \mathbb{N}$. Show that if $\{x_n\}$ is a CS so is $\{y_n\}$. Give an example to show that the converse may not hold.

(3) Define a sequence $\{x_n\}$ of rational numbers as,

$$x_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

(a) Show by induction that $x_n = 1 - \frac{1}{n+1}$. (Hint: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.)

(b) Show that $\{x_n\}$ is a CS.

(4) Let $0 \leq b_n \leq a_n$ be rational numbers. Show that if the sequence defined as, $x_n = \sum_{k=1}^{n} a_k$ is a CS, so is the sequence defined as $y_n = \sum_{k=1}^{n} b_k$. (Comparison Test).

(5) Show that the following sequences are CS.

(a) The sequence $\{z_n\}$ defined as,

$$z_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}.$$

(b) The sequence $\{u_n\}$ defined as,

$$u_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

(Hint: One way is to use the previous two problems).

(6) We consider the harmonic series, where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

(a) Show that for any $n \in \mathbb{N}$,

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \frac{1}{2}.$$

(b) Show that $\{x_n\}$ is not a CS.