Homework 11, Math 310, due 23rd November 2009

(1) Let \( \{x_n\} \) be a non-decreasing sequence. That is \( x_{n+1} \geq x_n \) for all \( n \). If the set \( \{x_1, x_2, \ldots, x_n, \ldots\} \) is bounded above show that \( \{x_n\} \) is a CS. (This is a very useful fact, since in general deciding whether a sequence is Cauchy is very difficult).

(2) Let \( a_n \leq b_n \leq c_n \) for all \( n \), where all these are rational numbers. Assume that \( \{a_n\}, \{c_n\} \) are CS and \( \{a_n\} \sim \{c_n\} \). Then show that \( \{b_n\} \) is a CS and \( \{a_n\} \sim \{b_n\} \). (This is what you studied in Calculus and called the squeeze theorem).

(3) Let \( \{x_n\} \) be a CS of rational numbers such that \( \{x_n\} \) is not related to the CS, \( 0 = \{0\} \) (that is the sequence with all terms zero, whose equivalence class is the zero element in \( \mathbb{R} \)).
   (a) Show that there exists a \( \delta > 0 \) and an \( N \in \mathbb{N} \) so that for all \( n \geq N \), \( |x_n| > \delta \).
   (b) Define a sequence \( \{y_n\} \) by \( y_n = 0 \) if \( n < N \) and \( y_n = x_n^{-1} \) for \( n \geq N \). (This makes sense since \( x_n \neq 0 \) if \( n \geq N \).)
       Show that \( \{y_n\} \) is a CS and \( [\{x_n\}][\{y_n\}] = 1 \).

(4) Let \( S \subset \mathbb{R} \) be an infinite bounded set. Show that there exists an infinite sequence \( \{x_n\} \) with \( x_n \in S \) for all \( n \) such that \( \{x_n\} \) is a CS. (Infinite sequence means, \( x_i \neq x_j \) if \( i \neq j \).)

(5) Use the above to prove Theorem 4.4 in the notes.

(6) Let \( I_n = [a_n, b_n] \) be closed intervals with \( a_n < b_n \) for all \( n \). Assume that \( I_n \subset I_{n-1} \) for all \( n \). Then show that \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \). That is there is an \( \alpha \) such that \( \alpha \in I_n \) for all \( n \). (This result is called the nested interval theorem).