Homework 11, Math 310, due 23rd November 2009

(1) Let $\{x_n\}$ be a non-decreasing sequence. That is $x_{n+1} \geq x_n$ for all n. If the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ is bounded above show that $\{x_n\}$ is a CS. (This is a very useful fact, since in general deciding whether a sequence is Cauchy is very difficult).

(2) Let $a_n \leq b_n \leq c_n$ for all n, where all these are rational numbers. Assume that $\{a_n\}, \{c_n\}$ are CS and $\{a_n\} \sim \{c_n\}$. Then show that $\{b_n\}$ is a CS and $\{a_n\} \sim \{b_n\}$. (This is what you studied in Calculus and called the squeeze theorem).

(3) Let $\{x_n\}$ be a CS of rational numbers such that $\{x_n\}$ is not related to the CS, $0 = \{0\}$ (that is the sequence with all terms zero, whose equivalence class is the zero element in $\mathbb{R}$).
   (a) Show that there exists a $\delta > 0$ and an $N \in \mathbb{N}$ so that for all $n \geq N$, $|x_n| > \delta$.
   (b) Define a sequence $\{y_n\}$ by $y_n = 0$ if $n < N$ and $y_n = x_n^{-1}$ for $n \geq N$. (This makes sense since $x_n \neq 0$ if $n \geq N$.) Show that $\{y_n\}$ is a CS and $[\{x_n\}][\{y_n\}] = 1$.

(4) Let $S \subset \mathbb{R}$ be an infinite bounded set. Show that there exists an infinite sequence $\{x_n\}$ with $x_n \in S$ for all n such that $\{x_n\}$ is a CS. (Infinite sequence means, $x_i \neq x_j$ if $i \neq j$.)

(5) Use the above to prove Theorem 4.4 in the notes.

(6) Let $I_n = [a_n, b_n]$ be closed intervals with $a_n < b_n$ for all n. Assume that $I_n \subset I_{n-1}$ for all n. Then show that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. That is there is an α such that $\alpha \in I_n$ for all n. (This result is called the nested interval theorem).