Homework 12, Math 310, due December 7th

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and as usual, for real numbers $a \leq b$ we denote by $[a, b]$, the closed interval consisting of real numbers x such that $a \leq x \leq b$.
  (a) Show that if $S, T \subset \mathbb{R}$ are bounded sets then so is $S \cup T$.
  (b) Show that for $a < b$, real numbers, $f([a, b]) = \{f(x) \mid x \in [a, b]\}$ is a bounded set. (Hint: Consider the set $S = \{x \in [a, b] \mid f([a, x])\text{is bounded}\}$).
  (c) Show that $f([a, b]) = [c, d]$ for some real numbers $c \leq d$. (Do not forget the intermediate value theorem).

(2) Show that there are infinitely many points (α, β) on the unit circle $x^2 + y^2 = 1$ with $\alpha, \beta \in \mathbb{Q}$. (Pythagorean triples).

(3) Consider the equation $x^2 - 2y^2 = 1$, called the Pell's equation. We wish to find integer solutions to this equation. It is clear that $x = 3, y = 2$ is such a solution.
  (a) Denote by $\mathbb{Z}[\sqrt{2}]$, the set of real numbers of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Z}$. Show that this set is closed under addition and multiplication.
  (b) If $a, b \in \mathbb{Z}$ with $(a + b\sqrt{2})(c + d\sqrt{2}) = 1$, where $c, d \in \mathbb{Z}$, show that $c = a, d = -b$ or $c = -a, d = b$.
  (c) If $a, b \in \mathbb{Z}$ and $(a + b\sqrt{2})(3 + 2\sqrt{2}) = c + d\sqrt{2}$, show that $c^2 - 2d^2 = a^2 - 2b^2$.
  (d) If we write $(3 + 2\sqrt{2})^n = a_n + b_n\sqrt{2}$ with $a_n, b_n \in \mathbb{Z}$, show that $a_n^2 - 2b_n^2 = 1$.