## Homework 4, Math 310, due Monday 28, September

Clarity and logical reasoning are paramount in these problems. Let a friend read your answers and comment on it to make sure that your arguments are complete.
(1) Give a direct proof to show that if $a, b$ are integers which are squares of integers, then $a b$ is the square of an integer.
(2) Write the converse to the statement appearing in the first problem. Is the converse true or false? If true, give a proof and if false, give a counterexample.
(3) Give a constructive proof to show that the equation, $x^{5}-x^{4}+$ $x^{3}-x^{2}+x-1=0$ has a solution in integers.
(4) Using results you have studied in previous courses you have taken, show that if $x, y$ are real numbers, then $|x y| \leq\left|x^{2}+y^{2}\right|$.
(5) Use intermediate value theorem (and thus a non-constructive proof) to show the following: A hiker walks up a hill starting at 6 am and reaching the top at 6 pm . He walks back next day from the top starting at 6 am and reaching the bottom at 6 pm along the same path. Show that there exists some point in his path where he was at the same time, going up or down.
(6) Show that if $a, b$ are postive real numbers and $a b \neq(a+b)^{2} / 4$, then $a \neq b$.
(7) (a) If $y$ is an irrational number and $x \neq 0$ is a rational number, show that $x y$ is irrational.
(b) If $x$ is a real number and $y$ is an irrational number, show that either $x+y$ or $-x+y$ is irrational.
(c) Give an example of $x, y$ as before, so that both $x+y$ and $-x+y$ are irrational.
(d) Give an example of $x, y$ as before so that one of $x+y,-x+y$ is irrational and the other is rational.
(8) If $a, b, c$ are integers with $a^{2}+b^{2}=c^{2}$, then show that either $a$ or $b$ is even.

