Homework 5, Math 310, due October 5th, Monday

(1) A variation of induction: Let $P(n)$ be predicates for all $n \in \mathbb{N}$. Assume that $P(1)$ is true. Also assume that for any $N \in \mathbb{N}$,

$$(P(n))(\forall n \leq N) \Rightarrow P(N+1).$$

Show that $P(n)$ is true for all $n \in \mathbb{N}$.

(2) Another variation: Let $P(n)$ be predicates, with $n \in \mathbb{N}$. Assume that $P(1)$ and $P(2)$ are true. Further assume that $P(n) \Rightarrow P(n+2)$. Then show that $P(n)$ is true for all $n \in \mathbb{N}$.

(3) I start with some elementary properties of division. Our universe is $\mathbb{Z}$.

(a) If $p, q, a \in \mathbb{Z}$ with $p \neq 0 \neq q$ and if $p \mid q$ and $q \mid a$ then $p \mid a$.

(b) If $0 \neq p \in \mathbb{Z}$ and $p \mid a$, $p \mid b$, then p divides any integer of the form $\alpha a + \beta b$ where $\alpha, \beta \in \mathbb{Z}$.

(c) If $N \neq 0 \neq M$ are integers and if $N \mid M$ and $M \mid N$, then $N = \pm M$. (The notation $N = \pm M$ is the short form to mean that $N = M$ or $N = -M$.)

All these can be proved easily from the definition of division.

Definition 1. *If $a, b \in \mathbb{Z}$ at least one of them non-zero, define the greatest common divisor of a and b denoted by* $\gcd(a,b)$ *as the natural number d such that d divides both a and b and if e is another integer dividing both a, b, then e divides d. For example* $\gcd(50, 75) = 25$.

(a) Let $a, b \in \mathbb{Z}$ with at least one of them non-zero. Show that the set, $S = \{pa + qb \in \mathbb{N} \mid p, q \in \mathbb{Z}\}$ is non-empty.

(b) Let d denote the minimal element of S, assured by induction. Show that d divides a and b using division algorithm.

(c) Show that $d = \gcd(a,b)$. In particular, $\gcd(a,b)$ exists.

(d) Assume that $a, b \in \mathbb{Z}$ with $a \neq 0 \neq b$. If $\gcd(a,b) = 1$, show that for any other integer c, $\gcd(ab, c) = \gcd(a, c) \cdot \gcd(b, c)$. Give an example where the last equation is false when $\gcd(a,b) \neq 1$.

(4) Prove that for any odd number $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^n}{n}\right) = 1.$$

(5) Prove that for any $n \in \mathbb{N}$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$