Homework 6, Math 310, due October 12th, Monday

(1) As usual, define \( n! = 1 \cdot 2 \cdot 3 \cdots n \) where \( n \) is any natural number. By convention \( 0! \) is defined to be 1. Similarly \( \binom{n}{r} \) for \( 0 \leq r \leq n \) is defined to be,

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

(a) Show that

\[
\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}
\]

(b) Prove the binomial theorem,

\[
(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n
\]

(2) Let \( Z \), the set of integers be the universal set. Write using roster method the following sets.

(a) \( A = \{ a \in Z \mid a^2 < 7 \} \).

(b) \( B = \{ a \in Z \mid \exists b \in Z, ab = 1 \} \).

(c) \( C = \{ a \in Z \mid a = 3b + 1, b \in Z \} \).

(3) Let the notation be as in the previous problem. Write using roster method the following sets.

(a) \( B \cup C \)

(b) \( A - C \)

(c) \( A \cap B \cap C \)

(4) Prove that for any three sets \( P, Q, R \subseteq \Omega \), where \( \Omega \) is the universal set, \( P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R) \) and \( (P \cup Q)^c = P^c \cap Q^c \).

(5) Let \( S_1, S_2, \ldots, S_n \) be finite sets.

(a) Show that \( \bigcup_{i=1}^{n} S_i \) is a finite set, where the above notation is the short form for \( S_1 \cup S_2 \cup \cdots \cup S_n \).

(b) Show that \(|\bigcup_{i=1}^{n} S_i| \leq \sum_{i=1}^{n} |S_i|\).

(c) If \( S_i \cap S_j = \emptyset \) for all \( i \neq j \), show that \(|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|\).