Homework 7, Math 310, due October 19th, Monday

(1) Prove the formula,

$$|\cup_{i=1}^n S_i| = \sum_{i=1}^n |S_i| - \sum_{1\le i<j\le n} |S_i \cap S_j| + \sum_{1\le i<j<k\le n} |S_i \cap S_j \cap S_k| - \cdots + (-1)^{n-1} |\cap_{i=1}^n S_i|.$$

(Hint: When notation as above seems confusing, always write out the expressions fully (without summation signs) for small values of n, till it is clear what is meant.)

(2) Recall the defintion of a prime number.

Definition 1. An element $p \in \mathbb{N}$ is a prime number if $p > 1$ and if $a \in \mathbb{N}$ divides p, then $a = 1$ or $a = p$.

(a) If $p \in N$ is a prime number and $a, b \in \mathbb{N}$ with p dividing ab, show that p divides a or p divides b.

(b) Show that any $n \in \mathbb{N}$, $n > 1$ can be written uniquely as a product of primes. That is, given such an n, there exists prime numbers $p_1, p_2, \ldots, p_m$ for some $m \in \mathbb{N}$ so that $n = p_1 p_2 \cdots p_m$ and this expression is unique upto reordering the p_is. (Hint: Use the version of induction $(P(n))(\forall n \le N) \Rightarrow P(N+1))$.

(3) Let $f : A \to A$ be a function. Define f^n to be composite $f \circ f \circ \cdots f$, n times. For example, $f^2 = f \circ f$ and $f^3 = f \circ f \circ f$. Show that if $f^n = \mathrm{Id}_A$, then f is bijective.

(4) Construct functions $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ for any natural number $n \ge 2$ so that $f_n^n = \mathrm{Id}_{\mathbb{R}^2}$, but $f_n^m \ne \mathrm{Id}_{\mathbb{R}^2}$ for $1 \le m < n$.

(5) If A, B are finite sets with $|A| = m \ge 1$ and $|B| = n \ge 1$, show that the set of all functions from $A \to B$, denoted by $\mathrm{Fun}(A, B)$ is a finite set and $|\mathrm{Fun}(A, B)| = n^m$.

(6) Let $\mathbb{R}[x]$ denote the set of all polynomials. Show that the function $d : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $d(f(x)) = \frac{df(x)}{dx}$ is surjective but not injective.