Homework 9, Math 310, November 9th

For these problems, you may use any of the stated properties of natural numbers, integers and rational numbers in the hand-out (both proved and merely stated but not proved). But, mention the property you are using in some recognizable way (for example: lemma such and such, or state it or...). You could also use any of the induction methods we have done in class or previous homeworks.

(1) Show that for any two rational numbers a, b, $|a-b| \geq ||a|-|b||$.

(2) Let $S \subset \mathbb{Z}$ be a non-empty subset which is bounded below. That is, there exists an integer M so that for all $s \in S$, $s \geq M$. Show that S has a least element. Give an example (with a reasonable argument) where similar statement fails for a subset of $\mathbb{Q}$.

(3) Show that if $a, b \in \mathbb{Z}$ with $ab = 0$, then a or b is zero.

(4) Use division algorithm and the usual definition of prime number (You should be able to prove the division algorithm from the properties we have stated, though you may assume it for this problem) show that given a prime number p and any $a \in \mathbb{N}$, there exists unique integers $a_0, a_1, \ldots, a_k$ for some k with $0 \leq a_i < p$ such that $a = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$. (This is usually called the p-adic expansion of a). What would happen if we tried to do the same for any $a \in \mathbb{Z}$?

(5) For any rational number r, define the set $S(r)$ to be, $S(r) = \{a \in \mathbb{N} \mid ar \in \mathbb{Z}\}$. Show that for rational numbers r, s, the intersection of $S(r)$ and $S(s)$ is not empty. Also, show that if $a \in S(r)$ and $b \in S(s)$, then $ab \in S(rs)$.

(6) Let $S(r)$ for a rational number r be defined as above. Show that there exists an $n \in \mathbb{N}$ (depending on r) such that $S(r) = \{an \mid a \in \mathbb{N}\}$. Define a relation on $\mathbb{Q}$ as follows. If $r, s \in \mathbb{Q}$, $r \sim s$ if $S(r) = S(s)$. Show that this an equivalence relation.