Homework 11, Math 310, due November 21st, 2011

The first problem should be submitted separately to me, which will not count for your homework grade.

(1) Let Σ_n be the set $\{1, 2, \ldots, n\} = \{k \in \mathbb{N} | k \leq n\}$.
 (a) If $a \in \Sigma_n$, prove that there is a bijection $\phi : \Sigma_n \to \Sigma_n$ such that $\phi(a) = n$.
 (b) If $f : \Sigma_n \to \Sigma_m$ is an injective map, prove that $n \leq m$.
 (c) Prove that $\mathcal{P}(\Sigma_n)$, the power set, is bijective to Σ_{2^n}.

(2) Decide whether the following are Cauchy sequences or not (with proofs).
 (a) $\{x_n\}$ with $x_n = n$.
 (b) $\{y_n\}$ with $y_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. (Hint: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$)
 (c) $\{z_n\}$ where $z_n = \sum_{k=1}^{n} \frac{1}{k^2}$. (Hint: For $k \geq 2$, show that $\frac{1}{k^2} \leq \frac{1}{(k-1)k}$ and use the previous part.)

(3) Define as usual, $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ (called n *factorial*) for any natural number n. By convention, we define $0! = 1$.
 (a) If $a \in \mathbb{Q}$ and $\epsilon > 0$ is a rational number, show that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\frac{a^n}{n!}| < \epsilon$. (Hint: Use the fact we showed in class, that given $x \in \mathbb{Q}$ with $|x| < 1$ and an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x^n| < \epsilon$.)
 (b) Prove that the sequence $\{x_n\}$ where $x_n = \frac{a^n}{n!}$ is a Cauchy sequence.
 (c) Prove that the sequence $\{y_n\}$ where $y_n = \sum_{k=1}^{n} x_n$ with x_n as in the previous part is a Cauchy sequence.

(4) Define as usual the *choose function* $\binom{n}{r}$ for $0 \leq r \leq n$ with $n \in \mathbb{N} \cup \{0\}$ as,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

 (a) Prove that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.
 (b) Prove (by induction) the binomial theorem for $a, b \in \mathbb{Q}$ and $n \in \mathbb{N} \cup \{0\}$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$