Homework 12, Math 310, due November 28th, 2011

The first problem should be submitted separately to me, which will not count for your homework grade. You may use all the relevant definitions and nothing else. The problems are mathematically trivial, so these are mainly exercises in proof writing. The last problem uses some elementary algebra.

(1) Let $\Gamma \subset A \times B$ for two sets A, B. If the first and second projections from Γ to A, B are bijective, prove that A, B are bijective.
(2) Let Γ_1, Γ_2 be subsets of $A \times A$ for a set A, defining equivalence relations on A. Prove that $\Gamma_1 \cap \Gamma_2$ defines an equivalence relation on A.
(3) Let $f : A \to B$ be a function and let $\Gamma \subset B \times B$ be an equivalence relation on B. Prove that the set $(f \times f)^{-1}(\Gamma) \subset A \times A$ (this can be described as $\{(a, a') \in A \times A | (f(a), f(a')) \in \Gamma\}$) is an equivalence relation on A.
(4) Let $a, b \in \mathbb{Z}$ with at least one of them non-zero and let $d = \gcd(a, b)$. From the definition of gcd we can write $a = a'd, b = b'd$ with $a', b' \in \mathbb{Z}$. Prove that $\gcd(a', b') = 1$.
(5) Let S be the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$. Define an operation $\oplus$ on S by, $(f \oplus g)(n) = f(n) + g(n)$ for any $f, g \in S$ and $n \in \mathbb{Z}$. Prove that, given $f \in S$ there exists a $g \in S$ such that $f \oplus g$ is the constant function zero. (That is, $(f \oplus g)(n) = 0$ for all $n \in \mathbb{Z}$).
(6) Use induction to prove that $\sum_{k=0}^{n} (2k + 1) = (n + 1)^2$.