Homework 13, Math 310, due December 5th, 2011

The first problem should be submitted separately to me, which will not count for your homework grade.

(1) Let a be a positive rational number. Give two different proofs for the fact $(1+a)^n \geq 1+na$ for any $n \in \mathbb{N}$, one using induction and the other using binomial theorem.

(2) Use the above to prove that given a rational number $a > 1$ and A any other rational number, there exists $N \in \mathbb{N}$ such that $a^N > A$.

(3) Let a be a fixed positive rational number. Choose (and fix) a natural number $M > a$.

  (a) For any $n \in \mathbb{N}$ with $n \geq M$, show that $\frac{a^n}{n!} \leq \frac{a^M}{M!}(\frac{a}{M})^{n-M}$.

  (b) Use the previous problem to show that, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{a^n}{n!} < \epsilon$.

  (c) We have proved in class that the sequence $\{x_n\}$ where $x_n = \sum_{k=0}^{n} b^n$ (the geometric series) is a Cauchy sequence if $|b| < 1$. Use this to prove (using the first part of this problem) that the sequence $\{y_n\}$ where $y_n = \sum_{k=0}^{n} \frac{a^k}{k!}$ is a Cauchy sequence. (As you have studied in Calculus, this sequence in the *limit* should give us e^a. We hope to define *limit* by end of this course.)