Homework 5, Math 310, due October 10th, 2011

You may use all that you know (from Calculus or other subjects), but do mention what you are using, by name of the result if it has one or stating briefly what it says.

(1) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is bijective.
(2) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x + 1$. Prove that for any $a \in \mathbb{R}$, the set $f^{-1}(a) = \{x \in \mathbb{R} | f(x) = a\}$ has either no elements (that is, it is empty) or one element or two elements.
(3) Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined as $f(m, n) = 2^{m-1}(2n-1)$. Show that f is a bijection.
(4) Let $f : A \to B, g : B \to C$ and $h : C \to D$ be functions. Show that $h \circ (g \circ f) = (h \circ g) \circ f$.
(5) Let $f : A \to A$ be a function. From the previous exercise (associativity), we can define $f^n = f \circ f \circ \cdots \circ f$, where we have taken $n - 1$ compositions. So $f^2 = f \circ f$, $f^3 = f \circ f \circ f$ etc. Prove that if f^n is injective for some $n \in \mathbb{N}$ then f is injective and if f^n is surjective for some $n \in \mathbb{N}$ then f is surjective.
(6) Here is a problem to think about. Do not submit. Can you show that there is no bijection $f : \mathbb{N} \to [0, 1]$, the closed interval in $\mathbb{R}$?