Homework 6, Math 310, due October 17th, 2011

(1) Let S_1, S_2, T be non-empty sets such that S_1 is bijective to S_2.
  (a) Show that $S_1 \times T$ is bijective to $S_2 \times T$.
  (b) Show that $\mathcal{P}(S_1)$ is bijective to $\mathcal{P}(S_2)$ (where $\mathcal{P}$ stands for power sets).
  (c) Show that $\text{Fun}(S_1, T)$ is bijective to $\text{Fun}(S_2, T)$ and $\text{Fun}(T, S_1)$ is bijective to $\text{Fun}(T, S_2)$ where $\text{Fun}(S, T)$ for two sets S, T is the set of all functions from S to T.

(2) Let $S_1 = \{a\}$ be a set consisting of just one element and let $S_2 = \{b, c\}$ be a set consisting of two elements.
  (a) Show that $S_1 \times \mathbb{Z}$ is bijective to $S_2 \times \mathbb{Z}$.
  (b) If T is any non-empty set, show that $\text{Fun}(T, S_1)$ is bijective to S_1 and $\text{Fun}(T, S_2)$ is bijective to $\mathcal{P}(T)$.

(3) Define a relation on $\mathbb{R}$ as follows. For $a, b \in \mathbb{R}$, $a \sim b$ if $a - b \in \mathbb{Z}$. Prove that this is an equivalence relation. Can you identify the set of equivalence classes with the unit circle in a natural way? (Hint: Trigonometric functions).

(4) Let $A = \mathbb{R}^2 - \{(0, 0)\}$. Define a relation on A by, for $a, b \in A$, $a \sim b$ if $b = \alpha a$ where α is a positive real number. (As usual, we write for $(x, y) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, $\alpha(x, y)$ to mean $(\alpha x, \alpha y)$). Prove that this is an equivalence relation. Can you identify the set of equivalence classes with the unit circle in a natural way?

(5) Let $\mathcal{C}(\mathbb{R})$ be the set of all continuous functions on $\mathbb{R}$. If $f, g \in \mathcal{C}(\mathbb{R})$ we say that $f \sim g$ if $\int_0^1 f dx = \int_0^1 g dx$. Show that this is an equivalence realtion. Can you identify the set of equivalence classes with a familiar set?

(6) Let $\mathcal{C}(\mathbb{R})$ be as in the previous problem and let $\mathcal{O}$ be the germs of continuous functions at the origin as defined in class. We have a natural surjective map $\pi : \mathcal{C}(\mathbb{R}) \to \mathcal{O}$. Define addition and multiplication in $\mathcal{O}$ by, $[f] \oplus [g] = [f + g], [f] \otimes [g] = [fg]$.
  (a) Prove that the above operations are well defined, as we illustrated in another situation. (This means, you must prove that if $f \sim F$ and $g \sim G$, then $f + g \sim F + G$ and $fg \sim FG$.)
  (b) If $f \in \mathcal{C}(\mathbb{R})$ with $f(0) \neq 0$, show that there exisits a $g \in \mathcal{C}(\mathbb{R})$ such that $[fg] = [1]$, where $[1]$ denotes the equivalence class containing the constant function 1.