Homework 8, Math 310, due October 31st, 2011

Several of the following are applications of the universal property of natural numbers. Remember that to apply it, you need a set, a function from the set to itself and an element in the set. So, most of the problems would be done if you can find these three things appropriately and you are expected to state these three explicitly before you appeal to the universal property.

(1) Let $x \in \mathbb{N}$. Show that there exists for each $n \in \mathbb{N}$ a natural number denoted by x^n (this is just a notation, but should tell you what we are doing) such that $x^1 = x$ and $x^{\sigma(n)} = x \cdot x^n$.

(2) Let S be a set and let $f : S \to S$ be a function. Show that for any $n \in \mathbb{N}$, there exists a function denoted by $f^n : S \to S$ such that $f^1 = f$ and $f^{\sigma(n)} = f \circ f^n$. Further, show that $f \circ f^n = f^n \circ f$ for all $n \in \mathbb{N}$.

(3) From the previous problem, we have $\sigma^n : \mathbb{N} \to \mathbb{N}$ for all $n \in \mathbb{N}$.
   (a) Show that for any $n \in \mathbb{N}$, $\sigma^{n+1}(\mathbb{N}) \subset \sigma^n(\mathbb{N})$, where we have used $n + 1$ for $\sigma(n)$ as we defined in class.
   (b) Show that the set $\sigma^n(\mathbb{N}) - \sigma^{n+1}(\mathbb{N}) = \{\sigma(n)\}$.
   (c) Let $\Sigma_n = \mathbb{N} - \sigma^n(\mathbb{N})$. Show that $\Sigma_{n+1}(\mathbb{N}) = \Sigma_n(\mathbb{N}) \cup \{n+1\}$ for all $n \in \mathbb{N}$.

(4) Show that for a set S, there exists an injective function $\phi : \mathbb{N} \to S$ if and only if there exists an injective, but non-surjective function $f : S \to S$. (Sets S satisfying this condition are called *infinite sets*.)