Homework 9, Math 310, due November 7th, 2011

(1) Given $M \in \mathbb{N}$, show that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $n^2 + n + 1 > M$.
(2) Check properties 5 and 11 of integers stated in the section on integers in the notes.
(3) Let $a_1, \ldots, a_n \in \mathbb{Z}$ be integers with at least one of them non-zero. Define $\gcd(a_1, \ldots, a_n)$ as the number $d \in \mathbb{N}$ such that $d|a_i$ for all i and if $e \in \mathbb{N}$ divides all the a_is, then $e|d$. Prove that $\gcd(a_1, \ldots, a_n)$ exists and is unique.
(4) Let $a, b \in \mathbb{Z}$, both non-zero. A natural number l is called the *lowest common multiple*, written $\mathrm{lcm}\,(a, b)$ if $a|l, b|l$ and if $e \in \mathbb{N}$ such that $a|e, b|e$, then $l|e$. Prove that $\mathrm{lcm}\,(a, b)$ exists and is unique. Further prove that $|ab| = \gcd(a, b)\mathrm{lcm}\,(a, b)$.
(5) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbb{Z}$. Prove that there exists an integer n such that $f(a) = an$ for all $a \in \mathbb{Z}$. (Hint: What is your guess for this mysterious n?)
(6) If $a, b \in \mathbb{Z}$, show that $|a - b| \geq ||a| - |b||$.
(7) If $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, show that $|a_0 + a_1 + \cdots + a_n| \leq |a_0| + |a_1| + \cdots + |a_n|$.