Homework 12, Math 310, due November 26th, 2012

(1) Let $q \in \mathbb{Q}$. Prove that there exists $a, b \in \mathbb{Z}, b \neq 0$ such that $q = a/b$ and $\gcd(a, b) = 1$.
(2) Prove that there is no rational number q such that $q^2 = 2$. (Hint: Assume there is and write $q = a/b$ as in the previous problem).
(3) Given $x < y$ with $x, y \in \mathbb{Q}$, prove that there exists a $z \in \mathbb{Q}$ with $x < z < y$.
(4) If $x > 1$ is a rational number and $y \in \mathbb{Q}$, prove that there exists an $N \in \mathbb{N}$ such that for all $n \geq N, n \in \mathbb{N}$, we have $x^n \geq y$.
(5) If $x \in \mathbb{Q}$ and $n \in \mathbb{N}$, prove that $(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}$.
(6) If $x > 0$ is a rational number, prove that there exists an $N \in \mathbb{N}$ such that for all $n \geq N, n \in \mathbb{N}$, we have $x > 1/n$.