Homework 123 Math 310, Do not submit, not graded

(1) Let A, B be any one of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ though we have not constructed it yet. We say a function $f : A \to B$ is *increasing* if for any $a, a' \in A$ with $a > a'$, we have $f(a) > f(a')$.
  (a) Prove that an increasing function as above is injective.
  (b) Construct an example of an injective function (from $A \to B$, A, B as above) which is not increasing.

(2) Let $f : \mathbb{N} \to \mathbb{Q}$ be a sequence and let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function. We say that the sequence $f \circ g : \mathbb{N} \to \mathbb{Q}$ a *subsequence* of the sequence given by f. Prove that if the sequence given by f is a Cauchy sequence, so is the one given by any subsequence.

(3) Let a be a fixed positive rational number. Choose (and fix) a natural number $M > a$. Define as usual, $n! = 1\cdot 2\cdot 3 \cdots (n-1)\cdot n$ (called n *factorial*) for any natural number n. By convention, we define $0! = 1$.
  (a) For any $n \in \mathbb{N}$ with $n \geq M$, show that $\frac{a^n}{n!} \leq \frac{a^M}{M!}(\frac{a}{M})^{n-M}$.
  (b) Using Problem 4 from Homework 12, show that, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{a^n}{n!} < \epsilon$.
  (c) We have proved in class that the sequence $\{x_n\}$ where $x_n = \sum_{k=0}^{n} b^n$ (the geometric series) is a Cauchy sequence if $|b| < 1$. Use this to prove (using the first part of this problem) that the sequence $\{y_n\}$ where $y_n = \sum_{k=0}^{n} \frac{a^k}{k!}$ is a Cauchy sequence. (As you have studied in Calculus, this sequence in the *limit* should give us e^a.)

(4) Define as usual the *choose function* $\binom{n}{r}$ for $0 \leq r \leq n$ with $n \in \mathbb{N} \cup \{0\}$ as,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

  (a) Prove that $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.
  (b) Prove (by induction) the binomial theorem for $a, b \in \mathbb{Q}$ and $n \in \mathbb{N} \cup \{0\}$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$