Homework 7, Math 310, due October 22nd, 2012

(1) Define a relation on $\mathbb{R}$ as follows. For $a, b \in \mathbb{R}$, $a \sim b$ if $a-b \in \mathbb{Z}$. Prove that this is an equivalence relation. Can you identify the set of equivalence classes with the unit circle in a natural way? (Hint: Trigonometric functions).

(2) Let $\mathcal{C}(\mathbb{R})$ be the set of all continuous functions on $\mathbb{R}$. If $f, g \in \mathcal{C}(\mathbb{R})$ we say that $f \sim g$ if $\int_0^1 f dx = \int_0^1 g dx$. Show that this is an equivalence relation. Can you identify the set of equivalence classes with a familiar set?

(3) Let $\mathcal{C}(\mathbb{R})$ be as in the previous problem. Define a relation on $\mathcal{C}(\mathbb{R})$ as follows. For $f, g \in \mathcal{C}(\mathbb{R})$, $f \sim g$ if there exists an open interval I containing $0 \in \mathbb{R}$ such that $f(x) = g(x)\, \forall x \in I$.
   (a) Prove that this is an equivalence relation.
   (b) Prove that if $f \sim F$ and $g \sim G$, then $f + g \sim F + G$ and $fg \sim FG$.
   (c) If $f \in \mathcal{C}(\mathbb{R})$ with $f(0) \neq 0$, show that there exists a $g \in \mathcal{C}(\mathbb{R})$ such that $fg \sim 1$, where 1 denotes the constant function 1. (The set of equivalence classes for this relation is usually denoted by $\mathcal{O}$ and called the *ring of germs of continuous functions* at the origin. Notice by part 2) above, it makes sense to add and multiply elements of $\mathcal{O}$.)

(4) Let P denote the set of all functions from a set S to the two element set $\{a, b\}$. Prove that there is a bijection from P to $\mathcal{P}(S)$, the power set of S.