Homework 9, Math 310, due November 5th, 2012

(1) For any $n \in \mathbb{N}$ consider the set $S_n \subset \mathbb{N}$ defined as $S_n = \{m \in \mathbb{N} | m^2 \geq n\}$.
   (a) Show that $S_n \neq \emptyset$ for any $n \in \mathbb{N}$.
   (b) Calculate the minimal element of S_{10}, assured by induction.

(2) Given $M \in \mathbb{N}$, show that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $n^2 + n + 1 > M$.

(3) Define a function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ by, $f(a, b) = (a+1, a+b+1)$. So, by universal property, we get a function $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ so that $\phi(1) = (1, 1)$ and $\phi(n + 1) = f(\phi(n))$. Let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the second projection, $\pi(a, b) = b$. Prove that for any $n \in \mathbb{N}$, $2\pi \circ \phi(n) = n(n + 1)$.

(4) Again, define $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ as $f(a, b) = (a+1, b+(a+1)^2)$. So, again we get a function $\phi : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ with $\phi(1) = (1, 1)$ and $\phi(n + 1) = f(\phi(n))$. Let π be as before. Show that $6\pi(\phi(n)) = n(n + 1)(2n + 1)$ for all $n \in \mathbb{N}$.

Here is a problem which you should not submit.

Given distinct points $P_1, P_2, \ldots, P_n$ in the plane such that the line joining any two contain a third from this collection, prove that they are all collinear.