Sample Problems

1. Set Theory

Let A, B, C, D be sets.

(1) Prove that if $A \subset B$ and $B \subset C$ then $A \subset C$.
(2) Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
(3) Prove that if $(A - B) \cap (A - C) = \emptyset$ then $A \subset B \cup C$.
(4) Prove that if $A \subset C, B \subset D$, then $A \times B \subset C \times D$.
(5) If $A \subset B$, prove that $\mathcal{P}(A) \subset \mathcal{P}(B)$, where $\mathcal{P}(A), \mathcal{P}(B)$ denote the power sets.

2. Equivalence Relations

We have two notations for equivalence relations on a set A. Sometimes we write $\sim$ for such a relation and sometimes we write it as $R \subset A \times A$.

(1) Check which of the following are equivalence relations and if it is an equivalence relation, decide how many distinct equivalence classes are there (possibly infinite).
   (a) Define a relation on $\mathbb{Z}$ by $a \sim b$ if 7 divides $a - b$.
   (b) Define a relation on $\mathbb{Z}$ by $a \sim b$ if $|a| = |b|$.
   (c) Define a relation on $\mathbb{Z}$ by $a \sim b$ if $a + b = 0$.
   (d) Define a relation on $\mathbb{Z}$ by $a \sim b$ if $ab \geq 0$.
(2) Describe the following equivalence relations on the set $A = \{1, 2, \ldots, 10\}$ as subsets $R \subset A \times A$.
   (a) $a \sim b$ if $a - b$ is even.
   (b) $a \sim b$ if $a - b$ is divisible by 3.
   (c) $a \sim b$ if $a^2 - a = b^2 - b$.
(3) Let A be as above. Describe **ALL** equivalence relations $R \subset A \times A$ in the following cases and calculate the number of distinct equivalence classes in each case.
   (a) $a \sim b$ if $a, b \leq 9$.
   (b) $a \sim b$ if $a + b = 11$.
   (c) $a \sim b$ if $a + b$ is even.

3. Functions

(1) Let $f : A \to B, g : B \to C$ be functions.
   (a) Prove that if f, g are injective, so is $g \circ f$.
   (b) Prove that if f, g are surjective, so is $g \circ f$.
   (c) Prove that if $g \circ f$ is injective, so is f and give an example in this situation when g may not be injective.

(d) Prove that if $g \circ f$ is surjective, so is g and give an example to show that in this situation f may not be surjective.

(2) Let $f : A \to B, g : C \to D$ be functions. Consider $F : A \times C \to B \times D$ given by $F(a, c) = (f(a), g(c))$. Prove that F is injective if and only if f, g are. Similarly, F is surjective if and only if f, g are.

(3) Let $A = \{a, b\}$ and $B = \{p, q, r\}$ be two sets with two (respectively three) distinct elements.
   (a) How many distinct functions are there from A to B?
   (b) How many of these are injective?
   (c) How many of these are surjective?

(4) Let $A = \{a, b, c\}, B = \{p, q, r\}$ be two sets with three distinct elements each. How many distinct functions are there from $A \to B$ which are bijective?

(5) Let A be a set and let $A_n \subset A$ for all $n \in \mathbb{N}$ with $A_n \neq A$ for all n. Assume that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} A_n = A$. Exhibit a function $f : A \to \mathbb{N}$ such that $A_n = \{a \in A | f(a) \leq n\}$ for all $n \in \mathbb{N}$.

4. Induction

(1) Prove that $2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$ for any $n \in \mathbb{N}$. The above will be usually written in the form $\sum_{k=1}^{n} 2k = n(n+1)$. (Notice that we have used $\cdots$ to show a pattern. This is common practice, but to be rigorous, but tedious, we will have to use the Universal property of $\mathbb{N}$, as we did in an earlier homework).

(2) Prove that $\sum_{k=1}^{n} 6k^2 = n(n + 1)(2n + 1)$ for any $n \in \mathbb{N}$.

(3) Prove that $\sum_{k=1}^{n} (2k - 1) = n^2$.

(4) Let T_n be a set with n distinct elements for $n \in \mathbb{N}$.
   (a) Prove that if there is an injective map $T_n \to T_m$, then $n \leq m$.
   (b) Prove that $T_n \times T_m$ has mn (distinct) elements.
   (c) Prove that $\mathcal{P}(T_n)$, the power set, has 2^n (distinct) elements.
   (d) Prove that the set of all functions from $T_n \to T_m$ has m^n (distinct) elements.