More Sample Problems

1. Peano's axioms and Natural numbers

(1) Write clear proofs for at least some of the unproven properties of addition and multiplication of natural numbers in the notes.
(2) Prove that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$ for any $n \in \mathbb{N}$.

2. Integers

(1) Prove the triangle inequality.
(2) Prove that for any integer x, $x^2 = |x|^2$.
(3) If d, e are positive integers with $\gcd(d, e) = 1$ and $a, b \in \mathbb{Z}$, prove that there exists an integer x such that $d|x - a, e|x - b$. (This is usually called the Chinese Remainder Theorem).
(4) Let $d > 0$ be an integer and $a \in \mathbb{Z}$. Writing $a = qd + r$ with $0 \leq r < d$ by division algorithm, show that $\gcd(a, d) = \gcd(r, d)$.
(5) Let $a, b, c \in \mathbb{Z}$ with $\gcd(a, b, c) = 1$ and $a^2 + b^2 = c^2$. Prove that one of a, b is odd and the other even.
(6) Let p be any prime and let a be a non-negative integer. Prove that there exists integers $a_0, \ldots, a_k$ (for some k) with $0 \leq a_i < p$ and $a = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k$. (This is called the p-adic expansion of a).

3. Rational Numbers

(1) Prove that there is no rational number q with $q^2 = p$ where p is a prime.
(2) Prove that there is no $q \in \mathbb{Q}$ such that $q^2 + 3q + 1 = 0$.
(3) If a, b are positive rational numbers, prove that $(a + b)^n \geq a^n + na^{n-1}b$.
(4) Prove that,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = 1 - \frac{1}{n+1}.$$

4. Cauchy sequences

(1) Below, $\{x_n\}$ will denote a sequence of rational numbers. Prove or disprove whether they are Cauchy sequences.
   (a) $x_n = \frac{1}{n^2}$.
   (b) $x_n = (-1)^n$.
   (c) $x_n = (-2)^{-n}$.
   (d) $x_n = \sum_{k=1}^{n} \frac{1}{2^k}$.

(2) For the following, fix a $q \in \mathbb{Q}$. Determine for what values of q are the sequences Cauchy and (if possible for what values are they not).
  (a) $x_n = q^n$.
  (b) $x_n = 1 + q + q^2 + \cdots + q^n$.
  (c) For all n, $|x_{n+1}| \leq |q||x_n|$.

(3) Let $\{x_n\}$ be a Cauchy sequence with $x_n \geq M > 0$ for all n. Prove that $\{\frac{1}{x_n}\}$ is a Cauchy sequence.