Math 417, Homework 10, due November 23rd 2010

(1) Let (X, d) be a metric space and let $A, B \subset X$ be two disjoint closed sets. Show that X is normal by using the function

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

(2) Let X be a topological space, $A \subset X$ a dense subset. Let $f : A \to Y$ be a continuous function where Y is regular. Further assume that if $\lim_{x \in A} x = x' \in X$ then $\lim_{x \in A} f(x)$ exists. Thus we can define a function $g : X \to Y$ by defining $g(x') = \lim_{x \in A, x \to x'} f(x)$. Show that g is continuous on X.

(3) Let X be a locally compact Hausdorff space, $x_0 \in X$ and $A \subset X$ closed with $x_0 \notin A$. Show that there exists a continuous function $f : X \to [0, 1]$ with $f(x_0) = 1$ and $f(A) = 0$. (Hint: One way is to imitate the proof of Urysohn's lemma). (Spaces which have such continuous functions for any point and a disjoint closed subset are called *completely regular*).

(4) Show that a compact metric space is second countable.

(5) Let $S_N \subset \mathbb{Q}$ for any $N \in N$ be defined as the set of all rational numbers $r = a/b$ with $a \in \mathbb{Z}, b \in \mathbb{N}$ and $b \leq N$. Show that there exists continuous functions $f_N : \mathbb{R} \to [0, 1]$ such that $f_N(r) = 0$ for all $r \in S_N$ and $f_n(\pi) = 1$. If there exists a function $f : \mathbb{R} \to [0, 1]$ such that $f(a) = \lim f_N(a)$ (the limit may not exist in general, but we are assuming that it does) show that f is not continuous.