Math 417, Homework 2, due 21st September 2010

(1) Let $f : A \to B$ be a function from the set A to the set B. Define a relation $a \sim a'$ for $a, a' \in A$ if $f(a) = f(a')$. Such a relation is called *relation associated to the function* f. Show that this is an equivalence relation. Conversely, if A is a set and $\sim$ an equivalence relation on A, show that there exists a set B and a function $f : A \to B$ so that the relation $\sim$ is the relation associated to f.

(2) Fix $n \in \mathbb{N}, n > 1$. Define a relation on $\mathbb{Z}$, $a \sim b$ if $a - b$ is an integer multiple of n. Show that $\sim$ is an equivalence relation. Show that $\mathbb{Z}/\sim$, the set of equivalence classes is a finite set and compute its cardinality.

(3) Let S be a set with an order relation $<_S$ (partial or total) and let $T \subset S$. Define a relation $t <_T u$ for $t, u \in T$ if $t <_S u$, called the *restriction*. Show that this is an order relation on T.

(4) Let A be a non-empty finite set. Show that the set of all bijections from A to itself is a finite set with cardinality $|A|!$ (where as usual ! denotes the factorial of a number).

(5) A real number α is called an *algebraic number* if there is a polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with $a_i \in \mathbb{Q}$ and $f(\alpha) = 0$. (For example, $\sqrt{2}, \sqrt{5}+1$). Let $A_n \subset \mathbb{R}$ be the set of all algebraic numbers, which are roots of polynomials as above of degree at most n. Show that A_n is countable. Thus show that the set of all algebraic numbers is countable. Deduce that there are non-algebraic real numbers. (For example, e, π are not algebraic, but difficult to prove this. Non-algebraic numbers are called *transcendental*).