Math 417, Homework 3, due 28th September 2010

(1) Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subset \mathcal{T}$ with the following property. For any $U \in \mathcal{T}$ with $x \in U$, there exists an element $U_x \in \mathcal{B}$ with $x \in U_x$ such that $U_x \subset U$. Show that any non-empty $U \in \mathcal{T}$ is the union of elements of $\mathcal{B}$ and thus $\mathcal{T}$ is the topology generated by $\mathcal{B}$. Such a set $\mathcal{B}$ is called a *base* for the topology $\mathcal{T}$.

(2) Let $\mathcal{B} \subset \mathcal{P}(\mathbb{R})$ be the set of all open intervals, $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. Show that $\mathcal{B}$ is a base for the topology generated by $\mathcal{B}$ (which is the topology we have used for $\mathbb{R}$).

(3) In $\operatorname{Spec} \mathbb{Z}$ with the Zariski topology, show that any closed set which is not $\operatorname{Spec} \mathbb{Z}$ is a finite set.

(4) Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and assume we give $X \times Y$ the product topology. Show that the collection of the sets of the form $U \times V$ where $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ form a basis for the product topology. Show that if $p_X : X \times Y \to X$ is the map $p_X(x, y) = x$ (called the projection) then for any open set W in $X \times Y$, $p_X(W)$ is open. (Such maps are called *open maps*).

(5) For a topological space $(X, \mathcal{T})$, show that it is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in the product topology.

(6) (a) Show that the series $\sum_{n=0}^{\infty} 10^{-n^2}$ converges. We will call the number that the series converges to by a.

(b) We have seen that the topology induced on $\mathbb{Z} \subset \mathbb{R}$ from the usual topology on $\mathbb{R}$ is discrete. Show that the topology induceds on the subset, $\mathbb{Z} + \mathbb{Z}a = \{x \in \mathbb{R} | x = m + na, m, n \in \mathbb{Z}\}$ is not discrete.