Math 417, Homework 4, due 5th October 2010

(1) Let $\mathbb{Q}$ be given the p-adic topology for a prime p and let U_n for an integer n as usual be the open set, $U_n = \{r \in \mathbb{Q} | v_p(r) \geq n\}$.
  (a) Show that if $r, s \in U_n$ then so is $r \pm s$. (That is U_n is a subgroup of $\mathbb{Q}$ with respect to addition).
  (b) Show that U_n is closed. (Again, this is true for any topological group-any open subgroup is closed).
  (c) Let $S = \{1, p, p^2, \ldots\} \subset \mathbb{Q}$. Find all limit points (if any) of S in $\mathbb{Q}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that it is continuous if and only if for any sequence $\{x_n\}$ with $x_n \in \mathbb{R}$ and $\lim x_n = x$, one has $\lim f(x_n) = f(x)$.

(3) We say that a subspace $A \subset X$ is dense in X if the closure of A is X. Show that if $f, g : X \to Y$,where Y is Hausdorff, are two continuous functions and $f_{|A} = g_{|A}$ (restrictions of f and g to A) where A is dense in X, then $f = g$.

(4) As usual, we identify the set of 2×2 matrices $M_2(\mathbb{R})$ with $\mathbb{R}^4$ by the map, $\phi : M_2(\mathbb{R}) \to \mathbb{R}^4$,

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d).$$

Show that this is a bijection (trivial). Thus, we may define a topology on $M_2(\mathbb{R})$ by declaring that a subset $U \subset M_2(\mathbb{R})$ is open if and only if $\phi(U)$ is open in $\mathbb{R}^4$.
  (a) Show that the map $M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R})$ given by matrix addition is continuous.
  (b) Show that the map $M_2(\mathbb{R}) \to \mathbb{R}$ given by $A \mapsto \det A$ is continuous.