**Math 417, Homework 5, due 12th October 2010**

(1) Let $(X, d_X), (Y, d_Y)$ be two metric spaces. Which of the following maps are metrics on $X \times Y$?
  (a) $D_1((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ where $x_i \in X$ and $y_i \in Y$.
  (b) $D_2((x_1, y_1), (x_2, y_2)) = \min\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$
  (c) $D_3((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$
  (d) $D_4((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) d_Y(y_1, y_2)$

(2) Let $X$ be the set of continuous functions on the closed interval $[0, 1]$. Define a function $d_{\text{int2}}$ on $X \times X$ as follows.

$$d_{\text{int2}}(f, g) = \left( \int_0^1 (f - g)^2 dx \right)^{1/2}.$$

Show that this is a metric on $X$. (The induced topology is called the $\ell^2$-topology). Can you compare the topologies induced by $d_{\text{int2}}$ with $d_{\text{int}}$ and $d_{\text{sup}}$ discussed in class?

(3) Let $f_n \in X$, where $X$ is as in the previous problem, be a seqeunce of functions defined by $f_n(x) = x^n$. It is clear that the sequence $\{f_n(x)\}$ of real numbers for any $x \in [0, 1]$ is convergent. Does the sequence $\{f_n\}$ converge in $X$ (to some function in $X$) in any of the three topologies we have defined?

(4) Show that if $d$ is a metric on $X$, then so are,

$$D_1(x, y) = \min\{d(x, y), 1\} \text{ and } D_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that the induced topologies for theses three metrics are the same. (Hint: If $0 \leq a \leq b$ then $\frac{a}{1+a} \leq \frac{b}{1+b}$)

(5) If $X$ is a topological space, we say that it is *first countable* if for any point $x \in X$, there exists a countable collection of neighbourhoods of $x$ and these collections as $x$ vary form a basis for the topology. Show that any metric space is first countable by exhibiting such a couuntable collection of open neighbourhoods for any point $x \in X$.