Math 417, Homework 5, due 19th October 2010

(1) (a) Let B_n be the open ball with center the origin of radius $a > 0$ in $\mathbb{R}^n$. Show that B_n is connected. If $n \geq 2$, show that $B_n - \{0\}$, the punctured ball is connected.
(b) Deduce that if $n \geq 2$, the complement of a finite set of points in $\mathbb{R}^n$ is connected.
(c) Show that the sphere $S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum x_i^2 = a^2\}$ is connected for $n \geq 2$.

(2) (a) Let $X = L^1([0,1])$ or $L^2([0,1])$ or $C^0([0,1])$ with the supremum topology. If $f \in X$, show that the map $\phi : \mathbb{R} \to X$ given by $\phi(t) = tf$ is continuous.
(b) Show that X is connected.

(3) We have seen in class that $\mathbb{Q}$ with the topology induced from $\mathbb{R}$ is not connected. Show that $\mathbb{Q}$ with the p-adic topology for any prime p is not connected.

(4) (a) Using Fundamental Thorem of Algebra (if you do not know it, please look it up), show that if $P(z)$ is a polynomial with complex coeffcients which is not constant, then the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto P(z)$ is onto.
(b) Let $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial with $a_i \in \mathbb{R}$ (or $\mathbb{C}$) and $n > 0$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is the map defined by $f(a) = P(a)$, then the inverse image of any bounded set is bounded. Thus inverse image of a compact set is compact.