Math 417, Homework 6, due 26th October 2010

We assume for the first problem that all spaces are Hausdorff and locally compact.

(1) (a) Show that a space X is compact if and only if the map $f : X \to \{point\}$ is proper.
   (b) Prove the converse of the theorem proved in class: A continuois map $f : X \to Y$ is proper if for any Z, the map $f \times \mathrm{Id} : X \times Z \to Y \times Z$ is closed. (Hint: Use Z to be the one point compactification of X)
   (c) We say that an infinite sequence $\{x_i\}$ with $x_i \in X$ escapes to infinity any compact susbet of X contains only finitely many of the x_i's. Show that a continuous map $f : \mathbb{R}^n \to Y$ is proper if and only if for any infinite sequence $\{x_i\} \subset \mathbb{R}^n$ escaping to infinity, the sequence $\{f(x_i)\} \subset Y$ is infinite and escapes to infinity.

(2) Show that $\mathbb{Q}$ with the usual topology is not locally compact.

(3) Let X be the set of continuous functions on the closed interval endowed with the supremum metric.
   (a) Show that X is locally compact if and only if the closed ball $B(0, c) = \{f \in X | d(f, 0) \leq c\}$ for any $c > 0$ where of course $d(f, 0) = \sup\{f(x) | x \in [0, 1]\}$ is compact.
   (b) Show that the sequnce of functions, $f_n(x) = x^n$ have no limit in X.
   (c) Deduce that X is not locally compact.