A quick survey of p-adic numbers

We fix a prime number p and define a map $v_p : \mathbb{Q} - \{0\} \to \mathbb{Z}$ as follows. If $0 \neq r \in \mathbb{Q}$, we may write $r = a/b$ with $a, b \in \mathbb{Z}$ and $a, b \neq 0$. Then there are unique non-negative integers $v_p(a), v_p(a)$ such that $p^{v_p(a)}$ divides a but $p^{v_p(a)+1}$ does not divide a. Similarly for b. Define $v_p(r) = v_p(a) - v_p(b)$. (Check that if we write $r = a/b = c/d$ with $a, b, c, d \in \mathbb{Z}$, then $v_p(a) - v_p(b) = v_p(c) - v_p(d)$ and thus $v_p(r)$ is well defined.)

We extend this to all of $\mathbb{Q}$ by declaring $v_p(0) = +\infty$. Here is an easy lemma.

Lemma 1. (1) *For any $r \in \mathbb{Q}$, $v_p(r) = v_p(-r)$*
(2) *If $r, s \in \mathbb{Q}$ then $v_p(rs) = v_p(r) + v_p(s)$*
(3) *For $0 \neq r \in \mathbb{Q}$, $v_p(r) \geq n$ for some n if and only if we can write $r = p^n a/b$ with $a, b \in \mathbb{Z}$ and p does not divide b.*
(4) *$v_p(r + s) \geq \min\{v_p(r), v_p(s)\}$.*

Define for any integer n, $A_n = \{r \in \mathbb{Q} | v_p(r) \geq n\}$ and for any $x \in \mathbb{Q}$, $A_n + x = \{a + x \mid a \in A_n\}$.

Lemma 2. (1) *If $a \in A_n$, so is $-a$.*
(2) *If $x, y \in A_n$, then so is $x \pm y$. (This follows from the last property in the above lemma [1].)*
(3) *$A_n \subset A_m$ if $n \geq m$.*
(4) *$A_n + x = A_n + z$ for any $z \in A_n + x$.*
(5) *$(A_n + x) \cap (A_m + y) = \emptyset$ or equal to $A_k + z$ where $k = \max\{m, n\}$ for some z.*

Now we are ready to define the p-adic toplology on $\mathbb{Q}$.

Lemma 3. *Let $\mathcal{T}$ be the subset of the power set of $\mathbb{Q}$ containing the empty set and arbitrary unions of the form $A_n + x$ for varying $n \in \mathbb{Z}$ and $x \in \mathbb{Q}$. Then $\mathcal{T}$ is the topology generated by the collection $\{A_n + x\}$.*

Proof. Since $A_n + x \in \mathcal{T}$, and any set in $\mathcal{T}$ is either empty or unions of $A_n + x$, we see that if we show $\mathcal{T}$ is a topology, we would have proved the lemma.

By choice $\emptyset \in \mathcal{T}$. Since $\mathbb{Q} = \cup_{x \in \mathbb{Q}} A_0 + x$, we see that $\mathbb{Q} \in \mathcal{T}$.

It is clear that union of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Finally, if $U_1, \ldots, U_m \in \mathcal{T}$, we wish to show that $\cap U_i \in \mathcal{T}$. If any of the $U_i = \emptyset$, then so is the intersection and we are done. So, let us assume that $U_i \neq \emptyset$ for all i. Then we can write $U_i = \cup(A_n + x)$. Thus, $\cap U_i = \cup((A_{n_1} + x_1) \cap \cdots \cap (A_{n_m} + x_m))$. If any of the intersections are empty, we may ignore those. If one of those is not empty, by lemma above, we have $(A_{n_1} + x_1) \cup \cdots (A_{n_m} + x_m) = A_k + z$ and thus $\cap U_i \in \mathcal{T}$. □

Finally we show that $\mathbb{Q}$ with the p-adic topolgy is Hausdorff. Let $x \neq y \in \mathbb{Q}$. One has $v_p(x-y) = n \in \mathbb{Z}$ (so $n \neq \infty$). I claim that $A_{n+1} + x \cap A_{n+1} + y = \emptyset$, which will prove that $(\mathbb{Q}, \mathcal{T})$ is Hausdorff. If not, we have $a + x = b + y$ for some $a, b \in A_{n+1}$. So $b - a = x - y$ and thus $v_p(b-a) = v_p(x-y) = n$. But, $a, b \in A_{n+1}$ implies by earlier lemma, $b - a \in A_{n+1}$ and thus $v_p(b-a) \geq n+1$. This is a contradiction.

Zariski Topology on $\operatorname{Spec} \mathbb{Z}$

Let $\operatorname{Spec} \mathbb{Z} = \{0, 2, 3, 5, \ldots, p, \ldots\}$, where p stands for a prime number. For any $0 \neq n \in \mathbb{Z}$, we define

$$\operatorname{Spec} \mathbb{Z}_n = \{a \in \operatorname{Spec} \mathbb{Z} | a \text{ does not divide } n\}$$

. (Some people define $\operatorname{Spec} \mathbb{Z}_0 = \emptyset$, assuming by convention that 0 does not divide 0). Let $\mathcal{T}$ be the set of all $\operatorname{Spec} \mathbb{Z}_n$ and the empty set. I calim that this is a topology on $\operatorname{Spec} \mathbb{Z}$. This follows from the foloowing easy lemma.

Lemma 4. (1) $\operatorname{Spec} \mathbb{Z}_n \cup \operatorname{Spec} \mathbb{Z}_m = \operatorname{Spec} \mathbb{Z}_d$ *where* $d = \gcd(n, m)$.
(2) $\operatorname{Spec} \mathbb{Z}_n \cap \operatorname{Spec} \mathbb{Z}_m = \operatorname{Spec} \mathbb{Z}_{nm}$.

Proof. For the first, let $a \in \operatorname{Spec} \mathbb{Z}_n \cup \operatorname{Spec} \mathbb{Z}_m$. Then $a \in \operatorname{Spec} \mathbb{Z}_n$ or $a \in \operatorname{Spec} \mathbb{Z}_m$, say $a \in \operatorname{Spec} \mathbb{Z}_n$. If $a = 0$, then clearly $a \in \operatorname{Spec} \mathbb{Z}_d$. If $a \neq 0$, then a is a prime not dividing n and since d divides n, a can not divide d. Thus $a \in \operatorname{Spec} \mathbb{Z}_d$. In the opposite direction, if $a \in \operatorname{Spec} \mathbb{Z}_d$, as before if $a = 0$, then $a \in \operatorname{Spec} \mathbb{Z}_n \cup \operatorname{Spec} \mathbb{Z}_m$. If $a \neq 0$, them a is a prime not dividing d. By the property of greatest common divisor, then a can not divide at least of n, m and then $a \in \operatorname{Spec} \mathbb{Z}_n$ or $a \in \operatorname{Spec} \mathbb{Z}_m$.

The second part is equally easy. □

We have seen in class that this topology is not Hausdorff. One way of seeing this is to note that any non-empty subset in $\mathcal{T}$ contains 0. If p is a prime number, for the topology to be Hausdorff, we must have open sets U_0, U_p which are neighbourhods of $0, p$ respectively and whose intersection is empty. But, $0 \in U_0$, being a neighbourhood of 0 and since $p \in U_p$, $U_p \neq \emptyset$ and thus contains 0. So $0 \in U_0 \cap U_p$.