De Rham Cohomology

1. Definition of De Rham Cohomology

Let X be an open subset of the plane. If we denote by $\mathcal{C}^0(X)$ the set of smooth (i. e. infinitely differentiable functions) on X and $\mathcal{C}^1(X)$, the smooth 1-forms on X (i. e. expressions of the form $fdx + gdy$ where $f, g \in \mathcal{C}^0(X)$), we have natural differntiation map $d : \mathcal{C}^0(X) \to \mathcal{C}^1(X)$ given by

$$f \mapsto \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

usually denoted by df. The kernel for this map (i. e. set of f with $df = 0$) is called the zeroth De Rham Cohomology of X and denoted by $H^0(X)$. It is clear that these are precisely the set of locally constant functions on X and it is a vector space over $\mathbb{R}$, whose dimension is precisley the number of connected components of X. The image of d is called the set of *exact* forms on X. The set of $pdx + qdy \in \mathcal{C}^1(X)$ such that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ are called *closed* forms. It is clear that exact forms and closed forms are vector spaces and any exact form is a closed form. The quotient vector space of closed forms modulo exact forms is called the first De Rham Cohomology and denoted by $H^1(X)$.

A *path* for this discussion would mean *piecewise smooth*. That is, if $\gamma : I \to X$ is a path (a continuous map), there exists a subdivision, $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\gamma(t)$ is continuously differentiable in the open intervals (t_i, t_{i+1}) for all i. Given a form ω and a path γ, we can integrate the form along the path.

Lemma 1. *If $\gamma(0) = P, \gamma(1) = Q$ and $\omega = df$, by fundamental theorem of calculus, we see that $\int_\gamma \omega = f(Q) - f(P)$.*

If γ is a closed path, we may think of γ as a map from I or S^1, whichever is convenient. Here is a *self-evident* lemma.

Lemma 2. *If $\gamma : S^1 \to \mathbb{R}^2$ is a closed path, then $Y = \mathbb{R}^2 - \gamma(S^1)$ has a unique unbounded connected component.*

Proof. Since $\gamma(S^1)$ is compact and hence bounded, we can find a closed bounded disc D containing $\gamma(S^1)$. It is immediate that $\mathbb{R}^2 - D$ is a connected open set contained in Y and hence contained in a connected component of Y. Any other connected component of Y must be hence completely contained in D and hence bounded. □

The union of the bounded connected components of Y as above is called the *open region* inside the closed curve $\gamma(S^1)$ and the complement of the unbounded component in $\mathbb{R}^2$ is called the *closed region* inside the closed curve $\gamma(S^1)$.

Lemma 3. *Let ω be a closed form on X. Then it is exact if and only if $\int_\gamma \omega = 0$ for all closed paths γ in X.*

Proof. If ω is exact, by lemma 1, we see that $\int_\gamma \omega = 0$. Conversely, given the vanishing, define a function on X by the following formula. Clearly we may assume that X is connected (and hence path connected). Fixing a point $a \in X$, for any $x \in X$, take a path γ from a to x and define $f(x) = \int_\gamma \omega$. The vanishing implies that $f(x)$ does not depend on the path γ and it is clear that $df = \omega$. □

2. Coboundary Homomorphism

Lemma 4 (partition of unity)**.** *Let X be covered by open sets $\{U_\alpha\}$. Then there exists a collection of smooth non-negative functions $\phi_\alpha : X \to \mathbb{R}$ such that* $\mathrm{Supp}\phi_\alpha \subset U_\alpha$*, the supports are locally finite and $\sum \phi_\alpha = 1$.*

Let $X = U \cup V$, union of two open sets. By partition of unity, we have $\phi_i, i = 1, 2$ such that $\mathrm{Supp}\phi_1 \subset U$ and $\mathrm{Supp}\phi_2 \subset V$, ϕ_i smooth on X and $\phi_1 + \phi_2 = 1$. If f is a smooth function on $U \cap V$, letting $f_1(x) = f(x)\phi_2(x)$ for $x \in U \cap V$ and $f_1(x) = 0$ for $x \in U - U \cap V$, we see that f_1 is smooth on U. Defining similarly, $f_2(x) = -\phi_1(x)f(x)$ for $x \in U \cap V$ and $f_2(x) = 0$ for $x \in V - U \cap V$, we see that $f_1 - f_2 = f$.

Now we define the coboundary map $H^0(U \cap V) \to H^1(X)$ as follows. Let $f \in H^0(U \cap V)$. Write $f = f_1 - f_2$ for smooth functions f_i on U, V as in the previous paragraph. Then $df_1 - df_2 = df = 0$, since f is locally constant and thus the two forms df_i patch together to get a form ω on X. Since it is locally exact, we see that $d\omega = 0$ and hence it is closed and thus defines an element in $H^1(X)$. Easy to check that this is well defined. So, we get,

$$\partial : H^0(U \cap V) \to H^1(X) \tag{1}$$

One can easily check that this map is a vector space homomorphim. That is, $\partial(f + g) = \partial(f) + \partial(g)$ and $\partial(af) = a\partial(f)$ for any real number a.

Lemma 5. $\partial(f) = 0$*, if and only if $f = f_1 - f_2$, where $f_1 \in H^0(U), f_2 \in H^0(V)$. The class of a closed form ω is in the image of ∂ if and only if $\omega_{|U}, \omega_{|V}$ are exact.*

Proof. If $f = f_1 - f_2$ with f_i locally constant, we have $df_i = 0$ and hence $\partial(f) = 0$. Conversely, if $\partial(f) = d\phi$ where ϕ is a smooth function on X (which is what we mean by a class is zero in $H^1(X)$), writing $f = f_1 - f_2$ as before, we see that $df_1 = d\phi_{|U}$ and $df_2 = d\phi_{|V}$ and thus letting $g_i = f_i - \phi$, we see that $dg_i = 0$ and hence $g_1 \in H^0(U), g_2 \in H^0(V)$ and $g_1 - g_2 = f$.

We have seen that if ω is in the image of ∂ then ω restricted to U, V are exact by our definition. Conversely, if $\omega_{|U} = df_1, \omega_{|V} = df_2$, then letting $f = f_1 - f_2$, we have $df = 0$ and hence $f \in H^0(U \cap V)$ and $\partial(f) = \omega$. □

3. Some computations

Lemma 6. *Let X be any of the following:*

1. $\mathbb{R}^2$.
2. *Open half planes, like $x > a$ or open quadrants like $x > a, y > b$.*
3. *Open rectangle or disc.*

Then $H^1(X) = 0$

Proof. If γ is a closed path in X, then the region enclosed by γ in $\mathbb{R}^2$ is completely contained in X and apply Green's theorem. □

Let $P = (x_0, y_0) \in \mathbb{R}^2$ and consider the form,

$$\omega_P = \frac{-(y - y_0)dx + (x - x_o)dx}{(x - x_0)^2 + (y - y_0)^2}.$$

Then ω_P is a smooth form everywhere except at P and it is closed. Letting $X = \mathbb{R}^2 - \{P\}$, we see that for any circle C around P, $\int_C \omega_P = 2\pi \neq 0$. Thus, by lemma

3, we see that $\omega_P \neq 0$ in $H^1(X)$. If ω is any other closed form on X, let $a = \int_C \omega$, and then letting $\omega' = \omega - \frac{a}{2\pi}\omega_P$, we have, ω' is a closed form with $\int_C \omega' = 0$. I claim, then that ω' is exact.

So, let ω be a closed form on X with $\int_C \omega = 0$. We wish to show that ω is exact. For ease of notation, let us assume that P is the origin. Then X is covered by the four open sets,

$$U_1 = \{x > 0\}, U_2 = \{y > 0\}, U_3 = \{x < 0\}, U_4 = \{y < 0\}.$$

By lemma 6, $\omega = df_i$ on U_i. Thus, $df_1 - df_2 = 0$ in $U_1 \cap U_2$, which is connected and hence we see that $f_2 = f_1 + c$ for some constant c. Since $df_2 = d(f_2 - c)$, it is clear that we may replace f_2 by $f_2 - c$ and hence assume that $f_2 = f_1$ in $U_1 \cap U_2$. Continuing, we may assume $f_3 = f_2$ in $U_2 \cap U_3$ and $f_4 = f_3$ in $U_3 \cap U_4$. Then we get, $f_4 = f_1 + c$ in $U_4 \cap U_1$ for some constant c.

Now cutting up our circle to be paths contained in U_i's and calculating the integral of ω with these f_i's, we see that $\int_C \omega = c$, which we have assumed to be zero. So, $f_4 = f_1$ in $U_4 \cap U_1$ and thus these f_i's patch up to get a smooth function ϕ on X and $d\phi = \omega$. Thus ω is zero in $H^1(X)$.

This shows that $H^1(X)$ is a one-dimensional vector space generated by the class of ω_P.

A similar argument will show that for any $P \neq Q \in \mathbb{R}^2$, $H^1(\mathbb{R}^2 - \{P, Q\})$ is a two dimensional vector space generated by ω_P, ω_Q.

The form ω_P and its integral is closely connected to winding numbers. Again, for convenience let us assume that P is the origin. If $\gamma : I \to \mathbb{R}^2 - \{0\}$ is a (smooth) path, we have defined the *winding number* $W(\gamma, 0)$ as follows. We can subdivide the plane into small regions of the form $a \leq \theta \leq b$ where $b - a < 2\pi$ and then we can divide I as $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\gamma([t_i, t_{i+1}])$ is completely contained in these chosen regions. Then the angle θ_i from $\gamma(t_i)$ to $\gamma(t_{i+1})$ is well defined and we define $W(\gamma, 0)$ to be the sum of these θ_i's. (Actually, we defined it by dividing this number by 2π.) One consequence is,

Lemma 7. *If γ is a path as above, then*

$$\int_\gamma \omega_0 = W(\gamma, 0).$$

One immediatley has the following corollary.

Corollary 8. *Let $A \subset \mathbb{R}^2$ be a closed connected set and let $P, Q \in A$. Then the class of ω_P, ω_Q are same in $H^1(\mathbb{R}^2 - A)$.*

Proof. Let γ be a closed path in $\mathbb{R}^2 - A$. Suffices to show that $\int_\gamma \omega_p = \int_\gamma \omega_Q$ by lemma 3. From the lemma above, suffices to show that $W(\gamma, P) = W(\gamma, Q)$. Reversing the roles, $W(\gamma, x)$ is a locally constant function on $\mathbb{R}^2 - \gamma$ and since P, Q are in the same connected component of this set, since A is connected, we see that $W(\gamma, P) = W(\gamma, Q)$. □

4. Important Consequences

Theorem 9. *Let $\phi : I \to \mathbb{R}^2$ be a homeomorphism to the image. Then $\mathbb{R}^2 - \phi(I)$ is connected.*

Proof. Let $Y = \phi(I)$ and assume that the complement is not connected. Fix points P, Q in different connected components of $\mathbb{R}^2 - Y$. Let $A = \phi([0, 1/2])$

and $B = \phi([1/2, 1])$ and let $S = \phi(1/2)$. Let $U = \mathbb{R}^2 - A, V = \mathbb{R}^2 - B$. Then $U \cap V = \mathbb{R}^2 - Y$ and $U \cup V = \mathbb{R}^2 - \{S\}$. We have the coboundary homomorphism,

$$\partial : H^0(U \cap V) \to H^1(U \cup V).$$

Since the H^1 is a one dimensional vector space generated by ω_S, for any $f \in H^0(U \cap V)$, $\partial(f) = a\omega_S$ for some $a \in \mathbb{R}$. By lemma 5, this means that $a\omega_S$ is exact on U, V. Any circle C of large radius around S is contained in both U, V. By lemma 3, we must have $\int_C a\omega_S = 0$, but this is just $2\pi a$. So, $a = 0$. In other words, the image of ∂ is zero.

Pick a locally constant function f on $U \cap V = \mathbb{R}^2 - Y$ such that $f(P) \neq f(Q)$, which is possible, since P, Q are in different connected componenets. $\partial(f) = 0$ implies by lemma 5 that there exists $f_1 \in H^0(U), f_2 \in H^0(V)$ such that $f_1 - f_2 = f$. But, then either $f_1(P) \neq f_1(Q)$ or $f_2(P) \neq f_2(Q)$. Since f_i's are locally constant, this means P, Q are in different connected components of U or V. Fixing one such, we see that P, Q are in different connected components of say $\mathbb{R}^2 - A$. Now call $A = Y_1$ and repeat the argument.

So, we get a sequence of closed intervals, $I \supset I_1 \supset I_2 \supset \cdots$ with length of $I_n = 2^{-n}$ and P, Q are in different connected components of $\mathbb{R}^2 - Y_n$, where $Y_n = \phi(I_n)$. By nested interval theorem, $\cap_{n=1}^{\infty} Y_n = \{T\}$. But $\mathbb{R}^2 - \{T\}$ is connected and so we can find a path connecting P, Q in this open set. So, there exists a small disc around T which does not intersect this path. It is immediate that Y_n for large n must be contained in this disc. So, the path does not intersect Y_n for large n and thus P, Q are in the same connected component of $\mathbb{R}^2 - Y_n$ for large n, contradicting our earlier assertion. This proves the theorem.

□

Theorem 10 (Jordan Curve Theorem). *Let $\phi : S^1 \to \mathbb{R}^2$ be a homeomorphism onto its image. Then $\mathbb{R}^2 - \phi(S^1)$ has exactly two connected components, one unbounded and the other bounded.*

Proof. The second part will follow from what we have already proved, if we prove the first part. Let $Y = \phi(S^1)$ and let $P \neq Q$ two points on Y. Then Y can be written as the union of two paths from P, Q, both homeomorphic to the unit interval. Call these A, B. Then $Y = A \cup B$ and let $U = \mathbb{R}^2 - A, V - \mathbb{R}^2 - B$. So, we have $U \cap V = \mathbb{R}^2 - Y$ and $U \cup V = \mathbb{R}^2 - \{P, Q\}$. We wish to show that $H^0(U \cap V)$ is two dimensional.

From the previous theorem, we know that $H^0(U), H^0(V)$ both are one-dimensional, consisting of the constant functions. If $f \in H^0(U \cap V)$ with $\partial(f) = 0$ by lemma 5, we can write $f = f_1 - f_2$ with f_i both constant functions on U, V respectively. Then it is clear that this kernel is one dimensional. For $f \in H^0(U \cap V)$, we can write $\partial(f) = a\omega_P + b\omega_Q$ for $a, b \in \mathbb{R}$. Again, by lemma 5, this form must be exact on U, V. Taking a large circle C containing Y, we see that,

$$\int_C a\omega_P + b\omega_Q = 2\pi(a + b).$$

Since this must be zero, we see that $a + b = 0$. Thus the image of ∂ is contained in the one dimensional vector space generated by $\omega = \omega_P - \omega_Q$. We will show that this is in the image and then we will have $H^0(U \cap V)$ to be two dimensional.

So, we want to show that ω restricted to both U, V are exact. But by corollary 8, this is clear. This finishes the proof. □