Math 418, Homework 10, due April 19th 2011

(1) Recall, for a continuous map $f : S^1 \to S^1$, we have the induced map $f_* : \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^1) = \mathbb{Z}$, which is muliplication by an integer d, called the degree of f.
   (a) Let f, g be continuous maps from the circle to itself. Show that $\deg(f \circ g) = \deg f \cdot \deg g$ and $\deg fg = \deg f + \deg g$ where $fg(z) = f(z)g(z)$, the multiplication on the right as complex numbers with absolute value 1. (Hint: Consider the map $(f, g) : S^1 \times S^1 \to S^1, (f, g)(z, w) = f(z)g(w)$.)
   (b) Show that, if f, g are as above, then f is homotopic to g if and only if $\deg f = \deg g$.
   (c) If $f(-P) = f(P)$ for all $P \in S^1$, show that d is even.

(2) Prove that if $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^m$, then $m = 2$.

(3) If $X \subset Y \subset Z$ are topological spaces with X a deformation retract of Y and Y a deformation retract of Z, show that X is a deformation retract of Z.

(4) Let $X = SL_2(\mathbb{R})$ be the group of 2×2 matrices with determinant 1 over $\mathbb{R}$. Show that $\pi_1(X)$ is non-trivial. (Hint: Consider $Y \subset X$, the set of matrices of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that Y is a retract of X).