Math 418, Homework 11, due April 26th 2011

(1) As usual, a *smooth path* is a smooth (i. e. infinitely differentiable) function $\gamma : I \to \mathbb{R}^2$, where γ is supposed to be defined and smooth in a suitable open interval containing I.

  (a) Let $\phi : I \to I$ be a smooth surjective map. Show that for any 1-form ω in a neighbourhood of $\gamma(I)$,
$$\int_{\gamma\circ\phi} \omega = \int_\gamma \omega.$$
  (Hint: Chain Rule)

  (b) Let $F : I \times I \to U \subset \mathbb{R}^2$ be a smooth map to an open set U. Assume that $F(0,s) = P, F(1,s) = Q$ for points $P, Q \in U$. (So this is a smooth path homotopy between $\gamma_0 = F(t,0)$ and $\gamma_1 = F(t,1)$). If ω is a closed form on U, show that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.

  (c) Let $F : I \times I \to U \subset \mathbb{R}^2$, as before, but now with $F(0,s) = F(1,s)$ for all $s \in I$. (So this is a homotopy of closed paths, but moving endpoints). With notation as before show that $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.

  (d) If $U = \mathbb{R}^2 - \{P\}$ and notation as before, show that $W(\gamma_1, P) = W(\gamma_2, P)$, the winding numbers. (Hint: Use ω_P).

(2) These are a few exercises on Winding numbers. If $\gamma : I \to \mathbb{R}^2 - \{0\}$ is a continuous path, we have defined the *winding number* $W(\gamma, 0)$ as follows. We can subdivide the plane into small regions (in polar coordinates) of the form $a \leq \theta \leq b$ where $b - a < 2\pi$ and then we can divide I as $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\gamma([t_i, t_{i+1}])$ is completely contained in these chosen regions. Then the angle θ_i from $\gamma(t_i)$ to $\gamma(t_{i+1})$ is well defined and we define $W(\gamma, 0)$ to be the sum of these θ_i's divided by 2π.

  (a) (Dog-on-a-leash) Let $\gamma, \delta : I \to \mathbb{R}^2 - \{P\}$be two closed paths so that the line segment joining $\gamma(t), \delta(t)$ does not contain P for any $t \in I$. Then show that $W(\gamma, P) = W(\delta, P)$. (Hint:Write an appropriate homotopy)

  (b) Let $\gamma : I \to \mathbb{R}^2$ be a closed path and let $P \in \mathbb{R}^2 - \gamma(I)$ be a point such that $\gamma(I)$ is contained in the half plane to the right of P. Show that $W(\gamma, P) = 0$.

  (c) For any path γ and a point $P \notin \gamma(I)$ and any other point Q, show that $W(\gamma, P) = W(\gamma + Q, P + Q)$, where $\gamma + Q$ is defined as, $\gamma(t) + Q$.

  (d) Show that, if γ is a closed path, the map $W(\gamma, P)$ for $P \notin \gamma$ is locally constant.