Math 418, Homework 4, due March 1st, 2011

(1) Show that a metric space is complete if and only if for every nested sequence $A_1 \supset A_2 \supset A_3 \supset \cdots$ of non-empty closed sets, with $\mathrm{diam} A_n \to 0$, $\cap A_n \neq \emptyset$.

(2) Let X be any topological space and Y a metric space. Giving $\mathcal{C}(X, Y)$ the set of continuous functions from X to Y the uniform metric, show that the evaluation map $e : X \times \mathcal{C}(X, Y) \to Y$, $e(x, f) = f(x)$ is continuous.

(3) Let X be a metric space and let $i : X \to Y, i' : X \to Y'$ be two completions. (Recall this means, Y, Y' are complete metric spaces, i, i' are isometric embeddings of X in Y, Y' and the images are dense). Show that there is an isometry $f : Y \to Y'$ so that $f \circ i = i'$.

(4) Let $X = \{0, 1, \ldots, p-1\}$ where p is a prime number.

(a) Given any integer $0 \leq a < p^{n+1}$, show that we can write $a = a_0 + a_1 p + + \cdots + a_n p^n$ with $a_i \in X$, uniquely.

(b) Consider the set $\mathbb{Z}_p = \prod_{n=0}^{\infty} X_n$ where $X_n = X$ for all n. If $x = \{x_n\}, y = \{y_n\} \in \mathbb{Z}_p$, define $d(x, y) = 0$ if $x = y$ and if $x \neq y$, $d(x, y) = p^{-m}$ where $x_i = y_i, i < m$ and $x_m \neq y_m$. Show that d is a metric on $\mathbb{Z}_p$.

(c) Consider the map $\phi : \mathbb{Z} \to \mathbb{Z}_p$ given as follows: If $0 \leq a < p^{n+1}$, define $\phi(a) = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$ where a_i's are defined as above. If $-p^{n+1} < a < 0$, define $\phi(a) = (b_0, b_1, \ldots, b_n, p-1, p-1, \ldots)$ where $b_0 + b_1 p + \cdots + b_n p^n = p^{n+1} + a$. Show that this is an isometric embedding, when we give $\mathbb{Z}$ the p-adic metric. (Recall the p-adic metric d_p on $\mathbb{Z}$ is given as follows. If $a \neq b \in \mathbb{Z}$, $d_p(a, b) = p^{-m}$ where p^m divides $a - b$ and p^{m+1} does not divide $a - b$. Of course, $d_p(a, a) = 0$).

(d) Show that $\phi(\mathbb{Z})$ is dense in $\mathbb{Z}_p$ and that $\mathbb{Z}_p$ is complete.