Math 418, Homework 5, due March 8th, 2011

(1) Let X, Y, Z be sets and let $\mathrm{Mor}(X, Y)$ etc. denote maps from X to Y. Given a map $f : X \times Z \to Y$ we get a map $\phi_f : Z \to \mathrm{Mor}(X, Y)$ as follows. $\phi_f(z)(x) = f(x \times z)$ where $x \in X, z \in Z$. If X, Y, Z are topological spaces and f is continuous, decide whether ϕ_f belongs to $C(Z, C(X, Y))$ in the following cases, where as usual C denotes continuous functions.
  (a) $C(X, Y)$ is given the topology of poitwise convergence.
  (b) Y is a metric space and $C(X, Y)$ is given the uniform metric.

(2) Let $\mathcal{F}$ be a collection of continuous functions from $\mathbb{R}$ to itself. Assume further that all functions in $\mathcal{F}$ are differentiable and given a point $a \in \mathbb{R}$, there exists an open neighbourhood U of a and a constant M such that $|f'(x)| \leq M$ for all $x \in U$ and all $f \in \mathcal{F}$. Show that $\mathcal{F}$ is equicontinuous.

(3) Let X be a compact space and let $f : X \to \mathbb{R}$ be a continuous non-negative function. Let g be the non-negative square root of f, which is clearly continuous. Show that there exists polynomials $P_n(t)$ such that $P_n(f)$ converges uniformly (in the sup metric) to g.

(4) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which is not identically zero, but $f(0) = f(1) = 0$. Consider the sequence of functions $g_n(x) = f(x^n)$. Show that $g_n(x)$ converges poitwise to the zero function, but does not converge uniformly (in the sup metric of course).

(5) Let X be a compact space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions which is non-increasing. That is, $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$. If f_n converges to a continuous function f pointwise, show that the convergence is uniform.