Math 418, Homework 5, due March 22nd 2011

(1) Show that the set $\mathcal{B}$ of bounded functions $\mathbb{R} \to \mathbb{R}$ is closed in the set of all functions $\mathbb{R} \to \mathbb{R}$ in the uniform topology but not in the topology of uniform convergence on compact sets.

(2) We will assume that $\mathcal{C}(X, Y)$, the set of continuous functions from X to Y is given the compact-open topology.
  (a) If f is in the closure of $B(K, U)$, where $K \subset X$ is compact and $U \subset Y$ open, then show that $f(K) \subset \overline{U}$.
  (b) Show that $\mathcal{C}(X, Y)$ is Hausdorff if Y is and it is regular if Y is.

(3) If $\mathcal{C}(X, \mathbb{R})$ is given the compact open topology, show that the addition map,

$$\mathcal{C}(X, \mathbb{R}) \times \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(X, \mathbb{R}), (f, g) \mapsto f + g$$

is continuous. (This essentially proves that $\mathcal{C}(X, \mathbb{R})$ is a toplogical group with respect to addition). Prove the same if $\mathbb{R}$ is replaced by S^1, the circle group. (Hint: It may be easier to use the fact that compact open topology coincides with uniform convergence on compact sets since $\mathbb{R}, S^1$ are metric spaces.)

(4) Let G be an abelian topological group and consider $\widehat{G}$, the set of all continuous homomorphisms from G to S^1 (remember S^1 is a group), with the compact open topology. Show that if G is compact then $\widehat{G}$ is discrete.

(5) Show that the collection $\{f_n\}$ where $f_n = x + \sin nx$ is pointwise bounded but not equicontinuous.