Math 418, Homework 8, due April 5th 2011

(1) Let $G = \{\omega \in \mathbb{C} | \omega^n = 1\}$.
   (a) Show that G is a group under multiplication.
   (b) Show that G acts on $S^1 \subset \mathbb{C}$ by, $z \mapsto \omega z$ for $z \in S^1, \omega \in G$.
   (c) Show that S^1/G is homeomorphic to S^1.
   (d) Show that the natural map $S^1 \to S^1 = S^1/G$ is just $z \mapsto z^n$.

(2) If G is a topological group, show that $\pi_1(G, e)$ is abelian, where e as usual is the identitiy element of G. (Hint: If $f, g : I \to G$ are loops at e, consider $H : I \times I \to G$, $H(s,t) = f(s) \cdot g(t)$.)

(3) If a group G acts on a topological space X, we say that G acts freely and properly discontinuously (a mouthful), if for any point $x \in X$, there is an open neighbourhood $x \in U$ such that $gU \cap hU = \emptyset$ if $g \neq h \in G$.
   (a) Show that if G acts on X freely and properly discontinuosly, the natural map $p : X \to X/G$ is a covering map.
   (b) Let G be the subgroup of homeomorphisms of the plane $\mathbb{R}^2$ generated by the two elements, $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (-x, y+1)$. Show that the action is free and properly discontinuous. (The quotient $\mathbb{R}^2/G$ can be identified with the Klein bottle).
   (c) If G is a finite group acting on a Hausdorff topological space, the action is free and proper discontinuous if and only if G has no fixed points. That is, if $gx = x$ for some $g \in G, x \in X$ then $g = e$.