# HOMEWORK 5, DUE THU MAR 4TH

*All solutions should be with proofs, you may quote from the book or from previous home works*

(1) Let $G$ be a finite group and let $p$ be the smallest prime dividing the order of $G$. Let $H$ be a subgroup of $G$ of index $p$. Show that $H$ is normal.

*Solution.* We let $G$ act on $G/H$, the set of left cosets, as follows. We define the map $T : G \to \text{Aut}(G/H)$ by, $T(g)(xH) = gxH$. (Check that $T(g)$ is indeed a bijection from $G/H$ to itself and thus gives an element of $\text{Aut}(G/H)$.) Next one checks $T$ is a group homomorphism and is straight forward. Let $K = \ker T$. If $g \notin H$, then $T(g)(eH = H) = gH \neq H$ and thus $g \notin K$. So, $K \subset H$ and thus $d = o(G/K)$ is divisible by $o(G/H) = p$.

Next, we see that since $K \subset G$, $d$ divides $o(G)$. Also, $K \subset \text{Aut}(G/H) = S_p$ and thus $d | o(S_p) = p!$. Thus $d$ divides $\gcd(o(G), p!) = p$. Since $p$ divides $d$ and $d$ divides $p$, $d = p$ and then $K = H$ and $K$ is normal, being kernel of a homomorphism. □

(2) Let $G$ be a group of order 231. Show that the 11-Sylow subgroup is in the center of $G$.

*Solution.* $231 = 11 \times 7 \times 3$. Since the number of 11-Sylow subgroups is $1 + 11k$ for some $k$ and divides 21, the only possibility is $k = 0$ and thus it is normal. Let $H$ denote this subgroup. Let $K$ be a 7-Sylow subgroup. Then, we get a homomorphism $T : K \to \text{Aut}(H)$, by conjugation, $T(g)(h) = ghg^{-1}$. But, $K \cong \mathbb{Z}/7\mathbb{Z}$ and $\text{Aut}(H)$ is a (cylic) group of order 10. So, $T$ must be trivial. Thus, the elements of $K$ commute with elements of $H$. Similar argument can be made for the 3-Sylow group and then it is easy to see that $H$ is in the center. □

(3) Let $G$ be a group of order $p^2q$, $p \neq q$ primes. Show that either a $p$-Sylow subgroup or $q$-Sylow subgroup is normal.

*Solution.* Assume that neither are normal. By the third Sylow theorem, we must have $1 + kp > 1$ $p$-Sylow subgroups and $i + kp$ should divide $q$. But $q$ is a prime, so $1 + kp = q$ and so $p|q - 1$. In particular $q > p$. Similarly we must have $1 + kq > 1$ $q$-Sylow subgroups and $1 + kq$ must divide $p^2$. So, $1 + kq = p$ or $1 + kq = p^2$. The first is impossible, since $q > p$. So, $1 + kq = p^2$. So, $q$ divides $p^2 - 1 = (p - 1)(p + 1)$. Thus $q$ must divide $p - 1$, which is not possibe, since $q \geq p + 1$ and thus it must divide $p + 1$ and so $q = p + 1$. The only such primes are $p = 2, q = 3$.

Thus we want to study groups of order 12. If the 3-Sylow subgroup is not normal, then there are $1 + 3k > 1$ of them dividing 4 and then this number has to be 4. Since these are cyclic groups of order 3, two of them can only intersect in identity. So, there are 8 elements of order 3 and its complement must be the unique 2-Sylow and hence normal. □

(4) Let $G$ be a group of order $pq$, $p < q$ primes.

(a) If $p$ does not divide $q - 1$, show that $G$ is cyclic.

*Solution.* Let $H$ be a $p$-Sylow subgroup and $K$ be a $q$-Sylow subgroup. Since there are $1 + kq$ $q$-Sylow subgroups and this number divides $p$, $k = 0$ and so $K$ is normal. Similarly, the hypothesis implies $H$ is normal. Also, the conjugation map $T : H \to \mathrm{Aut}(K)$ is a map from $\mathbb{Z}/p\mathbb{Z}$ to a (cyclic) group of order $q - 1$ and our assumption implies $T$ is trivial. So, elements of $H, K$ commute and then $G = HK$ is abelian. So, $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$, the last by Chinese remainder theorem. □

(b) If $p$ divides $q - 1$, show that there is a unique non-abelian group $G$ up to isomorphism.

*Solution.* Let $H, K$ be as before, Exactly as before, $K$ is normal. Thus, as before we get a homomorphism $T : H \to \mathrm{Aut}(K)$ and if this map is trivial, $G$ is abelian. So, assume $G$ is not abelian. Since $H \cong \mathbb{Z}/p\mathbb{Z}$, any homomorphism from $H$ must be either trivial or one-to-one (kernel of $T$ is a subgroup and $H$ has only trivial subgroups). Thus, since $\mathrm{Aut}(K)$ is a cyclic group of order $q - 1$, $T$ sends a generator $a$ to an element of order precisely $p$ of $\mathrm{Aut}(K)$ and $G$ is a semi-direct product of $H, K$ using $T$. □

(5) Let $\mathbb{F}_p$ as usual denote the field of $p$ elements (i. e. $\mathbb{Z}/p\mathbb{Z}$ for a prime $p$, where we have addition and multiplication as usual).

(a) Calculate the order of $GL(n, \mathbb{F}_p)$.

*Solution.* If $A \in G = GL(n, \mathbb{F}_p)$, we write it as $A = [\underline{a}_1, \ldots, \underline{a}_n]$, using column vectors. Then, $A \in G$ is equivalent to saying these vectors are linearly independent over $\mathbb{F}_p$. Thus, $\underline{a}_1$ can be any non-zero vector and so has a choice of $p^n - 1$ possibilities. Once we fix $\underline{a}_1$, $\underline{a}_2$ can not be a multiple of $\underline{a}_1$ and thus has a choice of $p^n - p$ possibilities. $\underline{a}_3$ can not be a linear combination of $\underline{a}_1, \underline{a}_2$ and thus has $p^n - p^2$ choices. Continuing, we see that

$$o(G) = (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1}).$$

□

(b) Find a $p$-Sylow subgroup (more or less explicitly describe).

*Solution.* From the previous part, the order of the the $p$-Sylow subgroup is $p^{\frac{n(n-1}{2}}$. Take $H \subset G$ to be the upper triangular matrices with 1 on the diagonal. I will leave you to check that this is indeed a subgroup, has the desired order and thus one such $p$-Sylow subgroup. (You may use facts learned in linear algebra.) □

(6) Let $G$ be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$ where $p$ divides $o(G)$.

(a) Prove that the $p$-Sylow subgroup of $G$ is normal.

*Solution.* Let $q = p^n | o(G)$ and $p^{n+1} \not| o(G)$. The condition immediately gives $(ab)^q = a^q b^q$ for all $a, b \in G$. So, the map defined by $T : G \to G$, $T(a) = a^q$ is a group homomorphism. Let $P$ be the kernel of $T$. Then $P$ is a normal subgroup of $G$ and every element $a$ with $a^q = e$ lies in $P$. Every element of every $p$-Sylow subgroup has this property, showing that $P$ must be the unique $p$-Sylow subgroup. □

(b) If $P$ is the $p$-Sylow subgroup, then there exists a normal subgroup $N$ such that $P \cap N = \{e\}$ and $PN = G$.

*Solution.* Take $N = T(G)$. Easy to check that $P \cap N = \{e\}$ and $PN = G$. □