MIDTERM, MATH 430, DUE THU MAR 15TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) If G is a finite abelian group with $n|o(G)$, show that number of solutions of $x^n = e$ in G is a multiple of n.

Solution. By the theorem proved in class, we can write $G = G_1 \times G_2 \times \cdots \times G_m$ where G_i is a cyclic group of order, say r_i. Then $o(G) = \prod r_i$ and thus $n| \prod r_i$. Then, it is elementary to see that $n| \prod \gcd(n, r_i)$. So, let $s_i = \gcd(n, r_i)$. Then, since $s_i|r_i$ and G_i is cyclic of order r_i, the set of solutions to $x^n = e$ wih $x \in G_i$ are precisely those $x \in G_i$ with $x^{s_i} = e$ (since $o(x)$ divides both n, r_i). These form a cyclic subgroup H_i of order s_i in G_i. Then, it is clear that the set of elements in G with $x^n = e$ is precisely $H_1 \times H_2 \times \cdots \times H_n$ and thus has order $\prod s_i$ and so n divides this number. □

(2) As usual, for a subgroup H of G, we write $N(H)$ to be the normalizer of H in G, $H = \{g \in G | gHg^{-1} \subset H\}$. If P is a p-Sylow subgroup of a finite group G, show that $N(N(P)) = N(P)$.

Solution. Since P is a normal subgroup of $N(P)$ it follows that P is the unique p-Sylow subgroup of $N(P)$. Now, let $g \in N(N(P))$ ($N(P) \subset N(N(P))$ is clear). Then, $gN(P)g^{-1} = N(P)$ and since gPg^{-1} is a p-Sylow subgroup and contained in $N(P)$, we see that $gPg^{-1} = P$ and then $g \in N(P)$. □

(3) If for an $a \in G$, G any group, one can solve the equation $x^2ax = a^{-1}$, show that $a = b^3$ for some $b \in G$.

Solution. We are given that there is an $x \in G$ such that,

$$x^2ax = a^{-1}. \tag{1}$$

Taking inverses, we get,

$$x^{-1}a^{-1}x^{-2} = a. \tag{2}$$

Substituting (2) in (1) for a, we get, $a^{-1} = x^2(x^{-1}a^{-1}x^{-2})x = xa^{-1}x^{-1}$, which says $ax = xa$ and so these commute. So, one becomes, $x^3 = a^{-2}$ and thus, $a^3x^3 = a$. So, $a = (ax)^3$. $\square$

(4) If G is a group of order 385, show that its 11-Sylow subgroup is normal and its 7-Sylow subgroup is in the center.

Solution. We use Sylow theorems. $385 = 5 \times 7 \times 11$. The number of 11-Sylow subgroups is 1 mod 11 and must divide 35 and thus must be 1. So, it is normal.

Similarly, the number of 7-Sylow subgroups is 1 mod 7 and must divide 55. Again, this forces it to be 1 and thus it is normal. Let $H = \mathbb{Z}/7\mathbb{Z}$ be the 7-Sylow subgroup. Let $Z(H) = \{g \in G | gh = hg \text{ for all } h \in H\}$. $Z(H)$ is a subgroup of G and contains H. If $K = \mathbb{Z}/11\mathbb{Z}$ is the 11-Sylow subgroup, we get a homomorphism $K \to \mathrm{Aut}(H)$ by conjugation action. But, $o(\mathrm{Aut}(H)) = 6$ and so there are no non-trivial such homomorphism and this says $K \subset Z(H)$ and a similar argument will show that all 5-Sylow subgroups are contained in $Z(H)$. Thus, 5, 7 and 11 divide $Z(H)$ and so $Z(H) = G$. So, H is in the center of G. $\square$

(5) Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, \sigma\}$, act on $\mathbb{F}_p v_1 + \mathbb{F}_p v_2$, a vector space of dimension 2 with basis v_1, v_2 where p is a prime, by $\sigma(v_1) = v_2, \sigma(v_2) = v_1$. Calculate the number of distinct orbits.

Solution. Since $o(G) = 2$, any orbit can have either one or two elements. We count them separately. If $x = a_i v_1 + a_2 v_2$, $\sigma(x) = a_1 v_2 + a_2 v_1$ and thus $\sigma(x) = x$ if and only if $a_1 = a_2$. Thus, the orbits with one element are precisely of the form $a(v_1 + v_2)$ where $a \in \mathbb{F}_p$ and so there are precisely p of them. The remaining $p^2 - p$ elements are paired into orbits containing 2 elements and thus there are $\frac{p^2-p}{2}$ of them. So, the total number of orbits is $p + \frac{p^2-p}{2}$. $\square$

(6) Calculate the number of distinct group homomorphisms from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$.

Solution. Any homomorphism $f : G \to A \times B$, where G, A, B are groups is just a pair of homomorphisms $f_1 : G \to A, f_2 : G \to B$ and $f(g) = (f_1(g), f_2(g))$. So, we calculate the number of homomorphisms to the two groups separately.

Any homomorphism from $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ must have the image contained the subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/10\mathbb{Z}$, since $\gcd(4,10) = 2$. Thus, this number is just the set of homomorphisms from $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and there are precisely 2 of them. Similar argument says the image of $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z}$ must be contained in $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/16\mathbb{Z}$ ($\gcd(4,16) = 4$) and the set of homomorphisms from $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ has cardinality 4. So, the number of homomorphisms $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$ is $2 \times 4 = 8$. $\square$