FINAL, DUE MAY 13TH

All solutions should be with proofs, you may quote from the book.

(1) Calculate the number of distinct group homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{C}^*$, the group of non-zero complex numbers under multiplication.

(2) Let $G = \mathbb{Z} \times \mathbb{Z}$ and let $f : A \to B$ be an onto homomorphism of abelian groups. Given a homomorphism $\phi : G \to B$, show that there exists a homomorphism $\psi : G \to A$ such that $f \circ \psi = \phi$. Give an example to show that this is false if we do not assume A is abelian.

(3) If $n_1, \ldots, n_r$ are positive integers and pairwise relatively prime (i.e. $\gcd(n_i, n_j) = 1$ if $i \neq j$), show that the commutative ring $(\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$ is isomorphic to $\mathbb{Z}/n_1 n_2 \cdots n_r\mathbb{Z}$.

(4) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

(5) Let $X^3 + aX + 1 \in \mathbb{Z}[X]$. Find for what values of $a \in \mathbb{Z}$ is this polynomial not irreducible over $\mathbb{Q}$.

(6) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find a $u \in K$ such that $K = \mathbb{Q}(u)$. (Caution: The u is not unique.)

(7) Let $L = D(t, u)$, the fraction field of the polynomial ring $D[t, u]$, in two variables where D is an infinite field of characteristic $p > 0$. Let $K = D(t^p, u^p) \subset L$. Show that there are infinitely many distinct subfields $M \subset L$ with $K \subset M$.

(8) Let p be a prime and let K be the splitting field of $X^p - 1$ over $\mathbb{Q}$. Determine the Galois group $G(K/\mathbb{Q})$.