HOMEWORK 1, DUE WED FEB 3RD

All solutions should be with proofs, you may quote from the book

(1) Decide which of the following are equivalence relations and describe the set of equivalence classes in a familiar form if it is an equivalence relation. (For example, in problem (b) below, the equivalence classes can be identified with $f(S)$, the image of f.)
   (a) Let $S = \mathbb{R}^2$ and If $p, q \in S$, we say $p \sim q$ if the distance between them is less than one.
   (b) Let $f : S \to T$ be a mapping. For $s_1, s_2 \in S$, we say $s_1 \sim s_2$ if $f(s_1) = f(s_2)$.
   (c) Let $S = \mathbb{R}$. We say for $a, b \in S$, $a \sim b$ if $a - b \in \mathbb{Z}$.
   (d) Let S be the set of non-zero complex numbers. If $a, b \in S$, $a \sim b$ if there is a positive real number r such that $a = rb$.

(2) Let S be a finite set of n elements and let $\mathcal{P}(S)$ be the power set (i.e. the set of all subsets of S). Show that it is finite and has 2^n elements. (In particular, there can not be a one-to-one, onto mapping from $S \to \mathcal{P}(S)$. The last statement is also true if S is infinite. Have you seen a proof?)

(3) Again, let S be a set with n elements. Construct a one-to- one correspondence $f : S \to S$ such that $f^n = \text{Id}$ (composition of f, n times), but $f^m \neq \text{Id}$ for $0 < m < n$.

(4) Again, let S be a set with n elements and $A(S)$, the set of all one-to-one onto maps from S to itself. Show that $A(S)$ has $n!$ elements.

(5) Let n, m be two positive integers. We will write $\mathbb{Z}/n\mathbb{Z}$ for J_n, used in the book, which is more standard. Let $\pi_n : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the map $\pi_n(a) = [a]$. Consider the map $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, $f(a) = (\pi_n(a), \pi_m(a))$. Find a necessary and sufficient condition on n, m so that f is onto.

(6) Let $\text{End}(\mathbb{Z}/n\mathbb{Z})$ (End is an abbreviation for *endomorphisms*) be the set of of all maps $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ satisfying $f([a] + [b]) = f([a]) + f([b])$ for all $a, b \in \mathbb{Z}$. Calculate the number of elements (cardinality) in this set.