HOMEWORK 3, DUE THU FEB 18TH

All solutions should be with proofs, you may quote from the book

(1) Let G be a group, H, K subgroups.
  (a) If H is normal, show that HK is a subgroup of G.

  (b) If H, K are both normal, show that HK is normal.

  (c) If H, K are both normal and $H \cap K = \{e\}$, show that for any $h \in H, k \in K$, $hk = kh$.

(2) (a) Let G be a group and H a subgroup of G. Define $N(H) = \{g \in G | gHg^{-1} = H\}$. Show that H is a normal subgroup of $N(H)$. ($N(H)$ is called the *Normalizer* of H in G.)

  (b) Let G be a group and let $Z(G) = \{g \in G | gx = xg$ for all $x \in G\}$, called the *center* of G. Show that $Z(G)$ is a normal subgroup of G.

  (c) Let $G = GL(n, \mathbb{R})$, the invertible $n \times n$ matrices. Describe $Z(G)$ explicitly.

(3) For a set S, we as usual denote the group $A(S)$, set of all one-to-one onto maps from S to itself, with composition as the group operation. Let G be a group and $f : G \to A(S)$ a group homomorphism. We shorten $f(g)(s)$ as just gs, when f is understood. (This is usually called an *action* of G on S.) We give below a few maps which you should decide whether are group homomorphisms and if so, find its kernel.
  (a) Consider the map $f : G \to A(G)$, given as $f(g) = \phi_g$ where $\phi_g(h) = gh$.

  (b) Consider $f : G \to A(G)$ given as $f(g) = \psi_g$ where $\psi_g(h) = ghg^{-1}$.

(c) Let H be a subgroup of G and let L be the left cosets of H in G. Let $f : G \to A(L)$ be defined as $f(g) = \theta_g$ where $\theta_g(aH) = gaH$.

(4) Let G, H, K be groups.

(a) Let $f : G \to H, g : G \to K$ be group homomorphisms. Show that the map $\phi : G \to H \times K$, $\phi(a) = (f(a), g(a))$ is a group homomorphism.

(b) Let $f : H \to G, g : K \to G$ be group homomorphisms. Show by an example that the map $\phi : H \times K \to G$ given by $\phi(a, b) = f(a)g(b)$ may not be a group homomorphism, but it is if G is abelian.

(c) Show that the map $f : G \to G$, $f(a) = a^{-1}$ may not be a group homomorphism, but it is if G is abelian.

(5) Let G be a group and $S \subset G$, a subset. We write $\hat{S} = \cap_{S \subset H} H$, intersection of all subgroups of G containing S.

(a) Let $S' = \{s^{-1} | s \in S\}$. Show that any element of the form $s_1 s_2 \cdots s_n$ for some n with $s_i \in S \cup S'$ is in $\hat{S}$ and conversely every element in $\hat{S}$ is of this form.

(b) Let $S = \{xyx^{-1}y^{-1} | x, y \in G\}$ (these elements are called *commutators*). Show that $\hat{S}$ (which is usually denoted by $[G, G]$, called the *commutator subgroup* of G) is a normal subgroup of G.

(c) Show that $G/\hat{S}$ is abelian.

(d) If H is any normal subgroup of G such that G/H is abelian, show that $\hat{S} \subset H$.

(6) (a) Let G be a group and Z its center. If G/Z is cyclic, show that $Z = G$.

(b) Show that any group of order 9 is abelian.