HOMEWORK 6, DUE THU MAR 11TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Let G be a finite abelian group of order n and let

$$G = \{g_1, g_2, \dots, g_n\}.$$

Let $g = \prod_{i=1}^{n} g_i$.

(a) Show that $g^2 = e$.

(b) If $o(G)$ is either odd or G has more than one element of order two, show that $g = e$.

(c) If G has exactly one element of order 2, say x, show that $g = x$.

(2) Let p be a prime number.

(a) Show that for any $x \in \mathbb{Z}$, $x^p \equiv x \bmod p$. (Fermat's little theorem)

(b) Show that $(p-1)! \equiv -1 \bmod p$. (Wilson's theorem)

(c) Assume p is odd. Write

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} = \frac{a}{b},$$

with $a, b \in \mathbb{Z}$. Show that $p|a$.

(3) Find all automorphisms of S_3.

(4) This is a long problem, but most cases are easy. Show that any group of order at most 30 is either of prime order or has a non-trivial normal subgroup, by analyzing each order. (In fact you should be able to do this for groups of order less than 60. We have seen A_5, whose order is 60, is simple.)

(5) Let $G = SL(2, \mathbb{F}_p)$, and Z be the center of $SL(2, \mathbb{F}_p)$. Let $P = PGL(2, \mathbb{F}_p) = SL(2, \mathbb{F}_p)/Z$, the projective linear group. Calculate $o(G)$ and $o(P)$.

(6) Let notation be as in the previous problem and assume that $p = 5$. Further assume that in this case, we know P is simple. We will as usual denote elements of $\mathbb{F}_p$ as $\{0, 1, 2, 3, 4\}$.

(a) Let
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$
Show that $\det A = \det B = 1$ and then we identify these with their images in P.

(b) Show that A, B generate a 2-Sylow subgroup H of P and $EHE^{-1} \neq H$, where ,
$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
So H is not normal.

(c) Let $C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Show that $\det C = 1$ and $o(C) = 3$. Show that $C \in N(H)$, the normalizer of H. Deduce that $o(N(H)) = 12$.

(d) Prove that $P \cong A_5$.