HOMEWORK 7, DUE THU MAR 25TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Let R, S be rings.

(a) Show that $A = R \times S$ is a ring with co-ordinate wise addition and multiplication. That is, $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bd)$. Show that the map $R \to R \times S$, given by $a \mapsto (a, 0)$ is a ring homomorphism. (Similarly for $S \to R \times S$. The construction can be done more generally, for a collection of rings. If R_i for $i \in I$, an indexing set, is a collection of rings, we can take $\prod R_i$ and give it as above a ring structure.)

(b) If R is a commutative ring with identity and $e \in R$ is an idempotent (that means $e^2 = e$), show that $1 - e$ is also an idempotent. Show that, $Re, R(1 - e)$ are subrings of R and $R = Re \times R(1 - e)$ as rings.

(c) Find all non-trivial idempotents (since $0, 1$ are always idempotents, we want to find others if any) in the rings $\mathbb{Z}/25\mathbb{Z}, \mathbb{Z}/15\mathbb{Z}$.

(2) Let k be a field and V a vector space (possibly infinite dimensional) over k.

(a) Show that $E = \{f : V \to V | f,\ k - \text{linear}\}$ is a ring with addition and multiplication defined as follows. $(f + g)(v) = f(v) + g(v)$ and $fg(v) = f(g(v))$. (If V is finite dimensional, you must recognize this as ring of square matrices, once we choose a basis).

(b) Take $V = k[X]$, polynomial ring in one variable. Show that we can identify X as an element of V, multiplication on V by X. Similarly $D = \frac{d}{dX}$, the derivative is an element of E. Show that $DX - XD = 1$, where 1 stands for the identity function.

(3) Let R be any *commutative* ring with identity. A map $D : R \to R$ is called a *derivation* if $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$. (This is called the Leibniz' rule or product rule in Calculus, if you remember).
   (a) Show that $D(1) = 0$.

   (b) Let $A = \{a \in R | D(a) = 0\}$ (often called the kernel of D). Show that A is a subring of R.

   (c) Assume that $\mathbb{Q}$, the field of rational numbers, is a subring of R. Then, show that $D(q) = 0$ for all $q \in \mathbb{Q}$.

   (d) Assume further, that for any element $a \in R$ there is an n, positive integer such that $D^n(a) = 0$ (D^n as usual is the short form for composition of D with itself n times) and that there is an $x \in R$ with $D(x) = 1$. Show that $R = A[x]$. That is, any element in R is just a polynomial in x with coefficients from A.

(4) Consider $R = M_2(\mathbb{R})$, the $n \times n$ matrices. We have seen that it is a (non-commutative) ring with the usual matrix addition and multiplication. So, we can multiply a matrix $A \in R$ with a vector $\mathbf{v} \in \mathbb{R}^2 = V$ as usual. (The results below are true for any $M_n(K)$, where K is any field and n is any positive integer, but the ideas can already be seen in the case $n = 2$.)
   (a) Let $\mathbf{0} \neq \mathbf{v} \in V$ and let $I = \{A \in R | A\mathbf{v} = \mathbf{0}\}$. Show that I is a left ideal of R.

   (b) Show that I is maximal. That is if $I \subset J \subset R$, where J is another left ideal, then $I = J$ or $J = R$.

   (c) Show that R has no non-trivial two sided ideals.

(5) These are a few problems on homomorphisms.
   (a) Let $A \in M_n(K) = R$, K any field and consider the map $\phi : K[X] \to R$, given by, $\phi(P(X)) = P(A)$ (this means, if $P(X) = a_0 + a_1 X + \cdots + a_r X^r$, $P(A) = a_0 I + a_1 A + \cdots + a_r A^r$). Show that this is a ring homomorphism. What is its kernel? (I am just asking for a word you might have learned in linear algebra).

(b) We define new binary operations on R as above. The addition is the same, but a new multiplication is given by $A \star B = BA$. Show that $(R, +, \star)$ is a ring which we call R^{op}. Show that the map $R \to R^{op}$ given by $A \mapsto A^T$ is a ring homomorphism.