HOMEWORK 9, DUE THU APR 8TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Let p be a prime number.
  (a) Show that the polynomial $x^n - p$ is irreducible in $\mathbb{Q}[x]$.

  (b) Show that $f(x) = \frac{x^p-1}{x-1} = 1 + x + \cdots + x^{p-1}$ is irreducible over the rationals. (Hint: Put $x = y + 1$ and use Eisenstein.)

  (c) Write $x^6 - 1$ as a product of irreducible polynomials in $\mathbb{Q}[x]$.

(2) Let $A = \mathbb{Z}[\sqrt{-5}]$.
  (a) Show that the only units in A are ± 1.

  (b) Show that $3, 2 + \sqrt{-5}$ and $2 - \sqrt{-5}$ are irreducible in A.

  (c) Prove that A is not a PID, using $3^2 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

(3) Let $A = \mathbb{C}[x, y]/I$ where I is the principal ideal generated by $y^2 - x^3 - x$. We also have an inclusion $B = \mathbb{C}[x] \subset A$ as a subring.
  (a) Show that $y^2 - x^3 - x$ is irreducible in $\mathbb{C}[x, y]$ and so, A is an integral domain.

  (b) Show that all maximal ideals of B are of the form $(x - a)B$ for some $a \in \mathbb{C}$. (Hint: Fundamental Theorem of Algebra).

  (c) Show that if $M \subset A$ is a maximal ideal of A, then $M \cap B$ is a maximal ideal of B.

(4) Let A be a PID.

(a) Let $R = K_1 \times K_2 \times \cdots \times K_n$, where K_is are fields, with the usual product ring structure. Let $a_1, \ldots, a_m \in R$ such that the ideal generated by these is the whole ring R. Show that we can find $q_2, q_3, \ldots, q_m \in R$ such that $a_1 + q_2 a_2 + q_3 a_3 + \cdots + q_m a_m$ is a unit in R.

(b) Let $a_1, \ldots, a_m \in A$ be such that $\gcd(a_1, \ldots, a_m) = 1$. Also assume that $m \geq 3$. Then show that we can find

$$p_2, \ldots, p_m, q_3, \ldots, q_m \in A$$

such that,

$$\gcd(a_1 + p_2 a_2 + \cdots + p_m a_m,\ a_2 + q_3 a_3 + \cdots + q_m a_m) = 1.$$

(c) Let $a_1, \ldots, a_m \in A$ with $\gcd(a_1, \ldots, a_m) = 1$. Show that we can find an invertible matrix U of size m so that,

$$(a_1, \ldots, a_m)U = (1, 0, \ldots, 0).$$

(Do this for $m \leq 3$, which has all the necessary ideas for full credit.)

(d) Using the above and imitating the proof we did in class, show that any torsion free finitely generated module over A is free.