Final Exam, Math 432, May 7th, 2005
Time 2hrs

Answer any five questions. If you answer more than five, indicate which five answers are to be graded. If you answer a question partially, it would be considered the whole question for grading purposes. Please write legibly.

1. (a) Define a finite Galois extension of fields and give an example of a finite extension which is not Galois.

   (b) Prove that any finite extension of finite fields is Galois and the Galois group is cyclic.

2. (a) State and prove Artin's theorem on linear independence of characters for a group.

   (b) If $K \subset L$ is a finite separable extension with $x_1, \ldots, x_n \in L$ a K-basis, show that there exists another basis $y_1, \ldots, y_n \in L$ such that $\mathrm{Tr}_{L/K}(x_i y_j) = \delta_{ij}$.

3. (a) Let $A \subset B$ be commutative rings with 1. Show that an element $b \in B$ is integral over A if and only if the ring $A[b] \subset B$ is a finitely generated A-module.

   (b) Show that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed and compute its integral closure.

4. (a) Let A be a commutative ring and M a finitely generated module. Let $N \subset M$ be a submodule and $I \subset A$ be an ideal. If $M = N + IM$, show that there exists an element $a \in A$, $a \equiv 1 \bmod I$ such that $aM \subset N$.

   (b) Recall that $\mathrm{Ann}M = \{a \in A \mid aM = 0\}$ where M is an A-module. Show that

   $$\mathrm{Ann}\, M + \mathrm{Ann}\, N \subset \mathrm{Ann}(\mathrm{Hom}_A(M, N)),$$

   for any two modules M, N.

5. (a) State some version of Hilbert's Nullstellensatz and describe all maximal ideals of $\mathbb{C}[X_1, \ldots, X_n]$ as explicitly as you can.

   (b) Find all (there are only finitely many) maximal ideals explicitly of the ring $\mathbb{C}[x, y]/(x^2 - y^3, y^2 - x^3)$.

6. (a) State the universal properties for $M \otimes N$, $S^n M, \wedge^n M$ for modules M, N over a commutative ring A.

   (b) Let $I = (x, y) \subset \mathbb{C}[x, y] = A$. Show that $I \subset \mathrm{Ann} \wedge^2 I$.

7. (a) State and prove Maschke's theorem.

   (b) Decide which finite subgroups of $\mathbb{C}^* \times \mathbb{C}^*$ have a faithful irreducible representation.

8. (a) Let $H \subset G$ and let χ_H be the character of a representation of H. Define the induced representation and compute its character in terms of χ_H.

   (b) Compute the character table for $G = S_3$ over $\mathbb{C}$. If V is the unique 2-dimensional irreducible representation of G, find irreducible modules W_i over G such that $V \otimes V = \oplus W_i$ as G-modules.