Homework 1, Math 5032, Due Jan 25th

1. Let $E = K(x)$ a simple algebraic extension of the field K with $[E : K] = p$, p a prime. Let $0 < m < p$ be an integer. Show that $E = k(x^m)$.

2. Let $f(X) \in K[x]$ be a polynomila of degree n and let E be its splitting field. Show that $[E : K]$ divides $n!$.

3. Let $f(X) = X^6 + X^3 + 1 \in \mathbb{Q}[X]$ and let E be its splitting field over $\mathbb{Q}$. Compute $[E : \mathbb{Q}]$.

4. Show that any element of a finite field can be written as a sum of two squares.

5. Let K be any field and let $E = K(x)$, the rational functions in one variable. (This means that E is the fraction field of $K[x]$, polynomial ring in x.) If L is a subfield of E properly containing K, show that $K(x)$ is algebraic over L. (Here is something to mull over: Let L be as before and let $\alpha \in L$ be an element of smallest positive degree, where degree is defined as follows. Write $\alpha = f/g$ with $f, g \in K[x]$ and define $\deg \alpha = \deg f - \deg g$. Then $L = K(\alpha)$. This is called Lüroth's Theorem.)

6. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements for a prime p. Let $\overline{\mathbb{F}}_q$ be its algebraic closure. Show that for any positive integer n there exists exactly one extension of $\mathbb{F}_q$ of degree n contained in $\overline{\mathbb{F}}_q$.