Homework 10, Math 5032, Due March 7th

A will denote a commutative ring with 1 and unless noted, $\otimes$ will mean $\otimes_A$.

1. If $M \to N$ is a surjective map of A-modules, show that the induced maps $T^n(M) \to T^n(N), S^n(M) \to S^n(N), \Lambda^n(M) \to \Lambda^n(N)$ are all surjective for all $n \geq 0$.

2. Let R be an A-algebra (this always means that we have a ring homomorphism $A \to R$ and the image of A is contained in the center of R) and let M be an A-module. Given an A-module homomorphism $M \to R$, show that we have a unique induced homomorphism of A-algebras, $T(M) = \oplus_{n=0}^{\infty} T^n(M) \to R$.

3. In the above, if R is commutative, show that we have an induced homomorphism of A-algebras $S(M) = \oplus_{n=0}^{\infty} S^n(M) \to R$.

4. If $I \subset A$ is an ideal, show that we have a surjective homomorphism of A-algebras $S(I) \to R(I) = \oplus_{n=0}^{\infty} I^n$. Give examples where this is an isomorphism and where it is not an isomorphism.

5. Let $K \subset L$ be a finite field extension. Show that L is separable over K if and only if $L \otimes_K L$ is a finite product of fields.

6. Let A be an integral domain. Show that A is a field if and only if all modules over A are A-flat.

7. Let K be a field and let V be a finite dimensional vector space with a non-singular symmetric bilnear form $f : V \times V \to K$. Let $\phi : V \to V^\vee$ be the induced isomorphism using f. For any basis $\{v_i\}$ of V, let $v_i' = \phi(v_i)$. Show that the element $\sum v_i \otimes v_i' \in V \otimes_K V^\vee$ is independent of the basis.